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ASYMPTOTIC ESTIMATES OF THE EIGENVALUES OF A SIXTH-ORDER BOUNDARY-VALUE PROBLEM OBTAINED BY USING GLOBAL PHASE-INTEGRAL METHODS

By P. BALDWIN

Department of Engineering Mathematics, Stephenson Building, University of Newcastle upon Tyne, Claremont Road, Newcastle upon Tyne NE1 7RU, U.K.

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CONTENTS

A boundary-value problem describing the onset of linear instability in a Bénard layer, is considered. The solutions of the sixth-order differential equation arising are expressed as Laplace integrals whose integrands involve a function satisfying a second-order equation with six transition points. W.K.B. approximations to this function, valid in regions associated with each transition point, are related by using global phase-integral methods. This allows solutions of the sixth-order problem to be estimated by steepest descents, and leads to an eigenvalue condition. The eigenvalue estimates are confirmed numerically by using the compound matrix method.

1. Introduction

When a Bénard layer has a non-uniform destabilizing steady-state temperature profile, convection sets in at a level where the local gradient sufficiently exceeds the adiabatic gradient for the inhibiting effects of viscosity and thermal conduction to be overcome. If this level is not at a boundary, the linearized equations governing the motion lead, in the Boussinesq approximation, to the boundary-value problem

$$(D^2-a^2)^3\,W+Ra^2(1-\zeta^2)\,W=0, \tag{1.1}$$

where
$$W\to 0 \quad \text{as} \quad \zeta\to\pm\infty, \tag{1.2}$$

(Baldwin 1987). Here, ζ is a dimensionless boundary-layer coordinate, W and a are a dimensionless vertical velocity and horizontal wavenumber respectively, R a Rayleigh number, and D denotes $\mathrm{d}/\mathrm{d}\zeta$. It is required to find the minimum value of the Rayleigh number for the onset of instability, and the associated wavenumber.

The problem (1.1)–(1.2) has some astrophysical interest as A-type stars are believed to have narrow convecting layers bounded by stable layers (Toomre *et al.* 1976). The method of solution is, however, of more general interest because it may be applied in principle to similar problems where the independent variable occurs quadratically in the coefficients of the differential equation.

The assumptions $Ra^2 \gg 1$ and $R/a^4 = O(1)$ allow the solutions of (1.1) to be expressed as Laplace integrals with an exponential exponent containing a large parameter, the rest of the integrand satisfying a second-order differential equation having six transition points and also involving the large parameter. This is described in §2 where details of the associated Stokes and anti-Stokes lines are then presented. In §3 global phase-integral methods (Heading 1977) are used to give approximations to solutions of the second-order equation throughout the transformed plane, but excluding small regions containing the transition points. The validity of the W.K.B., or phase-integral approximations, is discussed in §4 by using the theory of Olver (1974). Solutions to (1.1) are then considered in §5, where it is found that the Laplace-integral solutions are valid in restricted sectors only of the ζ-plane. To apply the boundary conditions (1.2), representations of the solutions are required across the ζ-plane, and these are found by relating the restricted solutions in ζ-sectors of common validity. The Laplace integrals are estimated by steepest descents and (1.2) applied to give the eigenvalue condition, which is then solved.

Finally, in §6, the estimated eigenvalues are used as initial estimates for an accurate numerical computation direct from the boundary-value problem (1.1)–(1.2). The compound matrix method, described for example by Drazin & Reid (1981), is used. It is found that even for the lowest mode, where the large parameter assumptions are certainly not valid, the estimates are surprisingly near the correct values, whereas for higher modes they are even better.

2. The Laplace integrals and the transformed differential equation

Experience of boundary-value problems similar to (1.1)–(1.2) suggests that the critical value of R for the onset of instability may be much larger than unity, and that a increases with R as the mode number of instability is increased (see, for example, Chandrasekhar (1961), p. 313; Duty & Reid (1964), p. 93). It is convenient to write

$$Ra^2 = \mu^6, \tag{2.1}$$

and

$$a = \mu\alpha, \tag{2.2}$$

in (1.1), where μ will be regarded as a large parameter. It has been shown by Baldwin (1987) that for an eigensolution of (1.1)–(1.2) with positive R, $R > a^4$ so that from (2.1) and (2.2)

$$0 < \alpha < 1. \tag{2.3}$$

On substitution for R and a from (2.1) and (2.2), the solutions of (1.1) may be written as the Laplace integrals

$$W(\mu\zeta, f; C) = \int_C \exp(\mu z\zeta) f(\mu, \alpha, z)\,\mathrm{d}z, \tag{2.4}$$

where $f(\mu, \alpha, z)$ is a solution of

$$(\mathrm{d}^2 f/\mathrm{d}z^2) - \mu^2[(z^2-\alpha^2)^3+1]f = 0, \tag{2.5}$$

and the contour C is chosen so that

$$[\exp(\mu z\zeta)\{\mu\zeta f - (\mathrm{d}f/\mathrm{d}z)\}]_C = 0. \tag{2.6}$$

The solution (2.4) may be estimated by the method of steepest descents if approximations to $f(\mu, \alpha, z)$ along the relevant paths of integration can be found. These are provided by the W.K.B. method applied to (2.5) for large μ. The W.K.B. approximations are normally valid in restricted regions only of the z-plane, and it is essential here to link them across the z-plane i.e. to use global phase-integral methods. The determination of approximations valid over such large areas of the z-plane is the main concern in this paper. To this end, the rest of this section is used to locate the transition points of (2.5) and to find the anti-Stokes lines associated with them.

Equation (2.5) has six transition points where $(z^2-\alpha^2)^3+1$ vanishes i.e. at

$$z = \pm z_k = \pm\sqrt{[\alpha^2 - \exp(-\tfrac{2}{3}k\pi\mathrm{i})]} \quad \text{for} \quad k = -1, 0, 1, \tag{2.7}$$

and they are distinct for α in the range (2.3). The convention

$$\arg z_k = \tfrac{1}{2}\arg[\alpha^2 - \exp(-\tfrac{2}{3}k\pi\mathrm{i})] \tag{2.8}$$

will be used, where the argument on the right of (2.8) has its principal value. To emphasize the symmetrical placing of the transition points about the real and imaginary axes it is convenient to adopt the notations $\bar{z}$, the complex conjugate of z, to denote the mirror image of z in the real axis, and z' to denote the mirror image of z in the imaginary axis. Hence, from (2.7)–(2.8),

$$\left.\begin{aligned} \bar{z}_0 &= -z_0 = -\mathrm{i}\sqrt{(1-\alpha^2)},\\ \bar{z}_1 &= z_{-1},\\ z_1' &= -z_{-1},\\ \text{and}\qquad \bar{z}_1' &= -z_1. \end{aligned}\right\} \tag{2.9}$$

This notation is used in figures 1–10 and tables 1–3.

In a region about $\pm z_k$, but excluding the immediate neighbourhood of $\pm z_k$, (2.5) has solutions which for large μ are asymptotic to the W.K.B. approximations associated with these points, i.e.

$$f \sim \frac{1}{[(z^2-\alpha^2)^3+1]^{\frac{1}{4}}}\exp\left\{\mu\int_{\pm z_k}^{z}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,\mathrm{d}z\right\}. \tag{2.10}$$

Associated with each transition point there are three Stokes lines

$$\mathscr{I}\int_{\pm z_k}^{z}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,\mathrm{d}z = 0, \tag{2.11}$$

approximated near $z = \pm z_k$ by the straight lines

$$\arg(z - [\pm z_k]) = \tfrac{2}{5}n\pi + \tfrac{4}{9}k\pi - \tfrac{1}{3}\arg(\pm z_k), \tag{2.12}$$

for integral n, and three anti-Stokes lines

$$\mathscr{R}\int_{\pm z_k}^{z}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,dz=0, \tag{2.13}$$

which near $z=\pm z_k$ bisect the angles between the lines (2.12).

For z on a Stokes line, it follows from (2.11) that

$$\mathscr{I}\int_{\pm \bar{z}_k}^{\bar{z}}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,dz=0,$$

and because from (2.9) $\pm\bar{z}_0=\pm z_0$, and $\pm\bar{z}_k=\pm z_{-k}$ for $k=\pm 1$, the Stokes lines have symmetry about the real axis. Also, because the integrand is even, it follows from (2.11) that

$$\mathscr{I}\int_{\mp z_k}^{-z}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,dz=0,$$

so that the Stokes lines have symmetry through the origin. Hence the Stokes lines, and similarly from (2.13) the anti-Stokes lines, have symmetry about both the real and imaginary axes.

Because $[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}$ has no finite singularities other than the branch points $z=\pm z_k$, all Stokes and anti-Stokes lines have one end at infinity unless they happen to connect transition points. Furthermore no two Stokes lines, and no two anti-Stokes lines can meet, other than at a transition point. Also from (2.11) and (2.13) the Stokes lines go to infinity in the directions

$$\arg z=\tfrac{1}{4}n\pi, \tag{2.14}$$

and the anti-Stokes lines in the directions

$$\arg z=\tfrac{1}{8}(2n-1)\,\pi, \tag{2.15}$$

for integral n.

Now because from (2.9) $z_0=\mathrm{i}\sqrt{(1-\alpha^2)}$, if $z=\mathrm{i}(\sqrt{(1-\alpha^2)}+y)$ where $y>0$,

$$\mathscr{I}\int_{z_0}^{z}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,dz=\mathscr{I}\int_{0}^{y}[(y^2+2y\sqrt{(1-\alpha^2)}+1)^3-1]^{\frac{1}{2}}\,dy=0,$$

so that the imaginary axis above z_0 is a Stokes line. Similarly, the imaginary axis between 0 and z_0 is an anti-Stokes line.

The positions of the Stokes and anti-Stokes lines in the first quadrant of the z-plane may be deduced by elementary arguments described in the Appendix, and use of symmetry then gives the results shown in figure 1. Continuous lines in this figure represent Stokes lines, and broken lines represent anti-Stokes lines. The wavy lines denote branch cuts associated with each transition point. It is necessary, in §3, when constructing tables 1–3, to label all the sectors created by the Stokes lines, anti-Stokes lines and branch cut at each transition point. These labels are also shown in figure 1 where the 'bar' and 'prime' notation used to describe the symmetrical positioning of the transition points (see (2.7)–(2.9)) is further exploited. The global positioning of the Stokes lines, other than helping to position the anti-Stokes lines, is not important in the subsequent analysis. It is noted that the structure of the anti-Stokes lines is unchanged for α in the range (2.3), but that there is a unique value $\alpha_1\approx 0.86$ of α in (2.3) for which a Stokes line joins z_0 and z_1. Figure 1 gives the position of the Stokes lines for $0<\alpha<\alpha_1$. As α increases to α_1 the Stokes line joining z_1 and z_1' moves up to go through z_0, merging with the lower two lines from z_0. For $\alpha>\alpha_1$, the Stokes line from z_1 emerges above

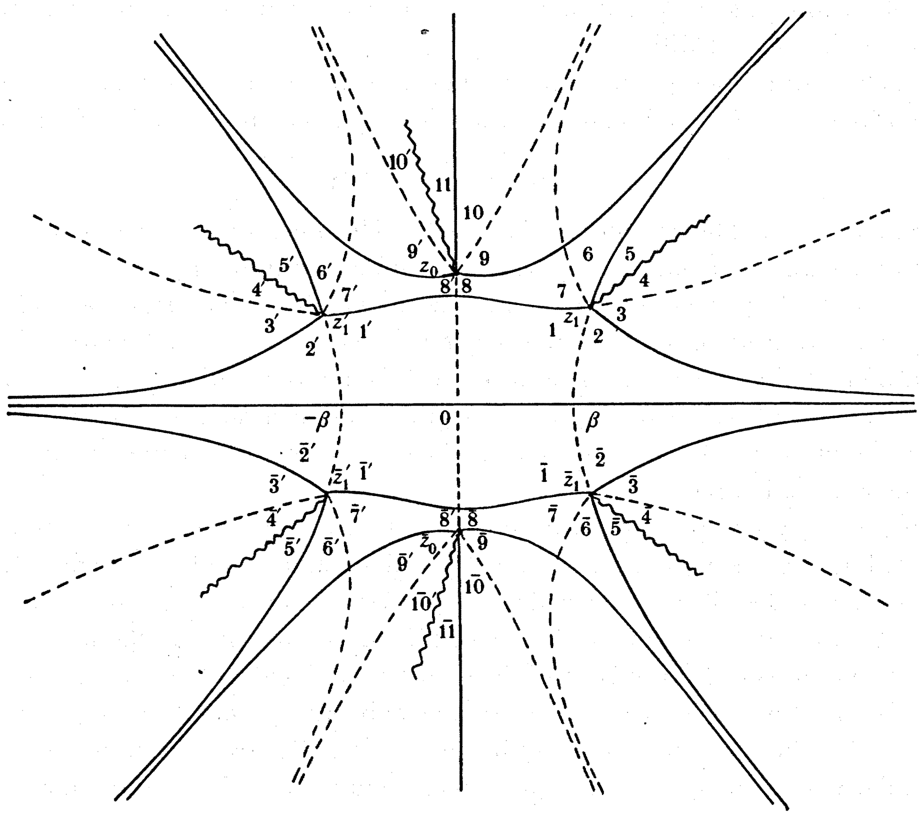

FIGURE 1. Stokes and anti-Stokes lines in the z-plane.

z_0, going to infinity in the direction $\arg z = \frac{1}{2}\pi$, whereas that from z_0 emerges below z_1, going to infinity in the direction $\arg z = 0$. This change in geometry is continuous, except for slopes at the transition points concerned.

3. THE APPLICATION OF GLOBAL PHASE-INTEGRAL METHODS

In this section, the W.K.B. approximations (2.10) associated with each transition point will be related by using global phase-integral methods, so that $f(\mu, \alpha, z)$ in (2.4) may be approximated throughout the z-plane.

The convenient notation of Heading (1962, 1977) will be used so that the W.K.B. approximations (2.10) are denoted by

$$(\pm z_k, z). \tag{3.1}$$

These are made precise when the root in the exponent is specified, the second solutions of (2.5) then being approximated by $(z, \pm z_k)$. If $(\pm z_k, z)$ is dominant i.e. exponentially large in some sector, this is denoted when appropriate by $(z, \pm z_k)_{\mathrm{d}}$. Similarly, subdominancy is denoted by a subscript s. It is also convenient to write

$$[a, b] = \exp\left\{\mu \int_a^b [(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,\mathrm{d}z\right\}, \tag{3.2}$$

for constants a and b so that

$$(z_1, z) = [z_1, z_0]\,(z_0, z), \tag{3.3}$$

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for example, expressing a W.K.B. approximation associated with z_1 in terms of a similar approximation associated with z_0.

Because all the transition points $\pm z_k$ are of order 1 i.e. are simple zeros of $(z^2-\alpha^2)^3+1$, all Stokes multipliers have value i, and if a branch cut is crossed in the anticlockwise direction

$$(\pm z_k, z) \rightarrow -\mathrm{i}(z, \pm z_k), \tag{3.4}$$

to preserve continuity. These results are discussed by Heading (1962) who shows how they may be used to trace an approximate solution locally round a transition point.

Heading (1977) has given sufficient conditions under which an approximate solution, valid locally round a transition point A, may be traced across to the neighbourhood of another transition point B and expressed in terms of W.K.B. approximations associated with B. He describes this process as tracing a phase-integral solution attached to A, across to B and attaching the solution to B. Relations such as (3.3) are used when reinterpreting the solution near the new transition point.

This process may always be carried out when the two transition points are joined by an anti-Stokes line, as are the pairs z_0 and $\bar{z}_0$, z_1 and $\bar{z}_1$, and z_1' and $\bar{z}_1'$ in figure 1. Sufficient conditions are also described under which the process is valid when the transition points are not joined by an anti-Stokes line, but these conditions are not met by the pair z_0 and z_1, for example, in that the configuration of anti-Stokes lines associated with them does not fully match that given by Heading. Even so, it will be assumed in this section that the process may be carried out for z_0 and z_1, and a justification will be given in §4. The same assumption will be made for the similarly situated pairs of transition points found by reflection in the real and imaginary axes.

Suppose that a solution of (2.5) is approximated near to z_1 on the anti-Stokes line joining z_1 and $\bar{z}_1$, but excluding the immediate vicinity of z_1, by $(z_1, z)+A(z, z_1)$ where A is a constant. Suppose further that the root of the exponent of (z_1, z) is chosen so that (z_1, z) becomes dominant is sector 1 of figure 1. The Stokes multipliers i and (3.4) may now be used to trace this solution locally round z_1 in an anticlockwise direction. The representations of this solution in each of the sectors 1–7 are listed in the left column of table 1. If $A = 0$, the solution subdominant in sectors 2 and 3 is represented locally round z_1, and a similar result is given for the solution subdominant in sectors 4, 5 and 6 if $A = \mathrm{i}$.

In sectors 1 and 7 the asymptotic approximation to the solution of (2.5) is $(z_1, z)_\mathrm{d}$. Under the assumption that this approximation remains valid as z moves away from z_1 towards z_0 and into the sectors 8 and 9 associated with z_0, (3.3) may be used to express this approximation in terms of a W.K.B. approximation valid in sectors 8 and 9 but excluding the immediate vicinity of z_0. When this is done (z_0, z) is subdominant in sectors 8 and 9, and may be traced locally round z_0 in an anticlockwise direction giving the representations in sectors 8–8′ shown in table 1 where

$$B = [z_1, z_0]. \tag{3.5}$$

Repeating this process from z_0 to z_1' with

$$C = B[z_0, z_1'], \tag{3.6}$$

gives the entries shown for sectors 1′ to 7′. These may be found by circling z_1' in an anticlockwise direction as was done for z_0 and z_1, or in a clockwise direction by reversing the sign of the Stokes multipliers and the sign of i in (3.4).

Table 1. Approximations near z_1 and their representations near other transition points

(Each entry is preceded by the label of the sector in which it is valid. A is an arbitrary constant.)

$B = [z_1, z_0] = \delta^{-1}\exp\{\mathrm{i}q\}$ $C = B[z_0, z_1'] = \delta^{-2}$

$D = [z_1, \bar{z}_1] = \exp\{-2\mathrm{i}p\}$ $E = A[\bar{z}_1, z_1] = A\exp\{2\mathrm{i}p\}$

$F = B[z_0, \bar{z}_0] = \delta^{-1}\exp\{-2\mathrm{i}p - \mathrm{i}q\}$ $G = C[z_1', \bar{z}_1'] = \delta^{-2}\exp\{-2\mathrm{i}p\}$

1.	$(z_1, z)_\mathrm{d} + A(z, z_1)_\mathrm{s}$	$\overline{1}$.	$D(\bar{z}_1, z)_\mathrm{d} + E(z, \bar{z}_1)_\mathrm{s}$
2.	$(z_1, z)_\mathrm{s} + A(z, z_1)_\mathrm{d}$	$\overline{2}$.	$D(\bar{z}_1, z)_\mathrm{s} + E(z, \bar{z}_1)_\mathrm{d}$
3.	$(1+\mathrm{i}A)(z_1, z)_\mathrm{s} + A(z, z_1)_\mathrm{d}$	$\overline{3}$.	$(D-\mathrm{i}E)(\bar{z}_1, z)_\mathrm{s} + E(z, \bar{z}_1)_\mathrm{d}$
4.	$(1+\mathrm{i}A)(z_1, z)_\mathrm{d} + A(z, z_1)_\mathrm{s}$	$\overline{4}$.	$(D-\mathrm{i}E)(\bar{z}_1, z)_\mathrm{d} + E(z, \bar{z}_1)_\mathrm{s}$
5.	$(A-\mathrm{i})(z, z_1)_\mathrm{d} - \mathrm{i}A(z_1, z)_\mathrm{s}$	$\overline{5}$.	$(E+\mathrm{i}D)(z, \bar{z}_1)_\mathrm{d} + \mathrm{i}E(\bar{z}_1, z)_\mathrm{s}$
6.	$(A-\mathrm{i})(z, z_1)_\mathrm{d} + (z_1, z)_\mathrm{s}$	$\overline{6}$.	$(E+\mathrm{i}D)(z, \bar{z}_1)_\mathrm{d} + D(\bar{z}_1, z)_\mathrm{s}$
7.	$(A-\mathrm{i})(z, z_1)_\mathrm{s} + (z_1, z)_\mathrm{d}$	$\overline{7}$.	$(E+\mathrm{i}D)(z, \bar{z}_1)_\mathrm{s} + D(\bar{z}_1, z)_\mathrm{d}$
8.	$B(z_0, z)_\mathrm{s}$	$\overline{8}$.	$F(\bar{z}_0, z)_\mathrm{s}$
9.	$B(z_0, z)_\mathrm{s}$	$\overline{9}$.	$F(\bar{z}_0, z)_\mathrm{s}$
10.	$B(z_0, z)_\mathrm{d}$	$\overline{10}$.	$F(\bar{z}_0, z)_\mathrm{d}$
11.	$B(z_0, z)_\mathrm{d} + \mathrm{i}B(z, z_0)_\mathrm{s}$	$\overline{11}$.	$F(\bar{z}_0, z)_\mathrm{d} - \mathrm{i}F(z, \bar{z}_0)_\mathrm{s}$
10′.	$-\mathrm{i}B(z, z_0)_\mathrm{d} + B(z_0, z)_\mathrm{s}$	$\overline{10}'$.	$\mathrm{i}F(z, \bar{z}_0)_\mathrm{d} + F(\bar{z}_0, z)_\mathrm{s}$
9′.	$-\mathrm{i}B(z, z_0)_\mathrm{s} + B(z_0, z)_\mathrm{d}$	$\overline{9}'$.	$\mathrm{i}F(z, \bar{z}_0)_\mathrm{s} + F(\bar{z}_0, z)_\mathrm{d}$
8′.	$B(z_0, z)_\mathrm{d}$	$\overline{8}'$.	$F(\bar{z}_0, z)_\mathrm{d}$
1′.	$C(z_1', z)_\mathrm{s}$	$\overline{1}'$.	$G(\bar{z}_1', z)_\mathrm{s}$
2′.	$C(z_1', z)_\mathrm{d}$	$\overline{2}'$.	$G(\bar{z}_1', z)_\mathrm{d}$
3′.	$C(z_1', z)_\mathrm{d} - \mathrm{i}C(z, z_1')_\mathrm{s}$	$\overline{3}'$.	$G(\bar{z}_1', z)_\mathrm{d} + \mathrm{i}G(z, \bar{z}_1')_\mathrm{s}$
4′.	$C(z_1', z)_\mathrm{s} - \mathrm{i}C(z, z_1')_\mathrm{d}$	$\overline{4}'$.	$G(\bar{z}_1', z)_\mathrm{s} + \mathrm{i}G(z, \bar{z}_1')_\mathrm{d}$
5′.	$\mathrm{i}C(z, z_1')_\mathrm{s} + C(z_1', z)_\mathrm{d}$	$\overline{5}'$.	$-\mathrm{i}G(z, \bar{z}_1')_\mathrm{s} + G(\bar{z}_1', z)_\mathrm{d}$
6′.	$C(z_1', z)_\mathrm{d}$	$\overline{6}'$.	$G(\bar{z}_1', z)_\mathrm{d}$
7′.	$C(z_1', z)_\mathrm{s}$	$\overline{7}'$.	$G(\bar{z}_1', z)_\mathrm{s}$

The W.K.B. approximations attached to z_1, z_0 and z_1' may now be traced across and attached to $\bar{z}_1$, $\bar{z}_0$ and $\bar{z}_1'$ respectively, via the relevant anti-Stokes lines. The approximation in sectors 1 and 2 becomes oscillatory on the anti-Stokes line separating them, and remains oscillatory along the anti-Stokes line into sectors $\overline{1}$ and $\overline{2}$. The dominancy of $(z_1, z)_\mathrm{d}$ in sector 1 is retained in $\overline{1}$, so that (3.3) with z_0 replaced by $\bar{z}_1$ gives

$$(z_1, z)_\mathrm{d} + A(z, z_1)_\mathrm{s} = [z_1, \bar{z}_1](\bar{z}_1, z)_\mathrm{d} + A[\bar{z}_1, z_1](z, \bar{z}_1)_\mathrm{s}.$$

This approximation may now be traced locally around $\bar{z}_1$, and use of the notation

$$D = [z_1, \bar{z}_1], \tag{3.7}$$

and

$$E = A[\bar{z}_1, z_1], \tag{3.8}$$

gives the entries shown in table 1 for sectors $\overline{1}$–$\overline{7}$. Similar representations near $\bar{z}_0$ and $\bar{z}_1'$ with

$$F = B[z_0, \bar{z}_0], \tag{3.9}$$

and

$$G = C[z_1', \bar{z}_1'], \tag{3.10}$$

complete table 1.

A check on the consistency of the results listed in table 1 may be made by tracing the approximations attached to $\bar{z}_1$ and $\bar{z}_0$ across to $\bar{z}_0$ and $\bar{z}_1'$ respectively. In sectors $\overline{1}$ and $\overline{7}$, the asymptotic approximation to the solution of (2.5) is $D(\bar{z}_1, z)_\mathrm{d}$, and in $\overline{8}'$ and $\overline{9}'$ the approximation is $F(\bar{z}_0, z)_d$. The former gives the approximation $D[\bar{z}_1, \bar{z}_0](\bar{z}_0, z)_s$ valid in sectors $\bar{8}$ and $\bar{9}$, and the latter $F[\bar{z}_0, \bar{z}_1'](\bar{z}_1', z)_s$ valid in $\bar{1}'$ and $\bar{7}'$. From (3.7) and then (3.5) and (3.9)

$$\begin{aligned} D[\bar{z}_1, \bar{z}_0](\bar{z}_0, z)_s &= [z_1, \bar{z}_0](\bar{z}_0, z)_s, \\ &= [z_1, z_0][z_0, \bar{z}_0](\bar{z}_0, z)_s, \\ &= F(\bar{z}_0, z)_s, \end{aligned}$$

as listed for $\bar{8}$ and $\bar{9}$. Similarly from (3.9) and then (3.6) and (3.10)

$$\begin{aligned} F[\bar{z}_0, \bar{z}_1'](\bar{z}_1', z)_s &= B[z_0, \bar{z}_1'](\bar{z}_1', z)_s, \\ &= B[z_0, z_1'][z_1', \bar{z}_1'](\bar{z}_1', z)_s, \\ &= G(\bar{z}_1', z)_s. \end{aligned}$$

as listed for $\bar{1}'$ and $\bar{7}'$.

Table 1 may now be used to approximate throughout the z-plane, except close to the transition points, that solution of (2.5) subdominant in sectors 2, 3, $\bar{2}$ and $\bar{3}$ (choose $A = 0$), subdominant in sectors 4, 5 and 6 (choose $A = \mathrm{i}$), or subdominant in $\bar{4}$, $\bar{5}$ and $\bar{6}$ (choose $D = \mathrm{i}E$, i.e. $A = -\mathrm{i}[z_1, \bar{z}_1]^2$ from (3.7) and (3.8)).

To complete the later analysis it will be necessary to deal similarly with solutions subdominant in all other sectors that may be extended to infinity.

Those sectors obtained by reflecting in the imaginary axis all those already dealt with i.e. 2–6, $\bar{2}$–$\bar{6}$ may be accounted for similarly by the method described above leading to table 1. The starting point is chosen by symmetry about the imaginary axis i.e. a solution of (2.5) is approximated near to z_1' on the anti-Stokes line joining z_1' and $\bar{z}_1'$, but excluding the immediate vicinity of z_1', by $(z, z_1') + A'(z_1', z)$, where A' is a constant. The order of the arguments in the W.K.B. approximations has been reversed as (z, z_1') here becomes dominant in sector 1′ as may be seen from the entry for this sector in table 1. The details are listed in table 2.

Alternatively it may be noted that figure 1 has symmetry through the imaginary axis apart from the cuts at z_0 and $\bar{z}_0$ that must initially be chosen not to coincide with Stokes or anti-Stokes lines. Hence representations of the solution of (2.5) in all sectors except 11 and $\overline{11}$ may be obtained directly from table 1 by

(*a*) priming all sectors and transition points;
(*b*) renaming the constants *A*–*G* by using a prime, say;
(*c*) changing the order of all arguments in (*, *) and [*, *]; and
(*d*) changing the sign of i in all coefficients.

The reason for (*d*) is that the tracing of solutions locally round transition points changes direction on reflection in the imaginary axis. In (*a*) it should be remembered that $(z_1')' = z_1$ and $(1')' = 1$ for example, because a prime denotes reflection in the imaginary axis for both sectors and transition points.

Table 3 deals with a solution subdominant in sectors 10, 11 and 10′. A table dealing with a solution subdominant in the conjugate sectors $\overline{10}$, $\overline{11}$ and $\overline{10}'$ may be found from table 3 by conjugating all sectors and numbers except z.

The coefficients of the W.K.B. approximations in tables 1–3 may all be expressed in terms

TABLE 2. APPROXIMATIONS NEAR z_1' AND THEIR REPRESENTATIONS NEAR OTHER TRANSITION POINTS

(Each entry is preceded by the label of the sector in which it is valid. A' is an arbitrary constant.)

$B' = [z_0, z_1'] = \delta^{-1}\exp\{-\mathrm{i}q\}$	$C' = B'[z_1, z_0] = \delta^{-2}$
$D' = [\bar{z}_1, z_1'] = \exp\{2\mathrm{i}p\}$	$E' = A'[z_1', \bar{z}_1'] = A'\exp\{-2\mathrm{i}p\}$
$F' = B'[\bar{z}_0, z_0] = \delta^{-1}\exp\{2\mathrm{i}p+\mathrm{i}q\}$	$G' = C'[\bar{z}_1, z_1] = \delta^{-2}\exp\{2\mathrm{i}p\}$
$1'$. $(z,z_1')_\mathrm{d}+A'(z_1',z)_\mathrm{s}$	$\bar{1}'$. $D'(z,\bar{z}_1')_\mathrm{d}+E'(\bar{z}_1',z)_\mathrm{s}$
$2'$. $(z,z_1')_\mathrm{s}+A'(z_1',z)_\mathrm{d}$	$\bar{2}'$. $D'(z,\bar{z}_1')_\mathrm{s}+E'(\bar{z}_1',z)_\mathrm{d}$
$3'$. $(1-\mathrm{i}A')(z,z_1')_\mathrm{s}+A'(z_1',z)_\mathrm{d}$	$\bar{3}'$. $(D'+\mathrm{i}E')(z,\bar{z}_1')_\mathrm{s}+E'(\bar{z}_1',z)_\mathrm{d}$
$4'$. $(1-\mathrm{i}A')(z,z_1')_\mathrm{d}+A'(z_1',z)_\mathrm{s}$	$\bar{4}'$. $(D'+\mathrm{i}E')(z,\bar{z}_1')_\mathrm{d}+E'(\bar{z}_1',z)_\mathrm{s}$
$5'$. $(A'+\mathrm{i})(z_1',z)_\mathrm{d}+\mathrm{i}A'(z,z_1')_\mathrm{s}$	$\bar{5}'$. $(E'-\mathrm{i}D')(\bar{z}_1',z)_\mathrm{d}-\mathrm{i}E'(z,\bar{z}_1')_\mathrm{s}$
$6'$. $(A'+\mathrm{i})(z_1',z)_\mathrm{d}+(z,z_1')_\mathrm{s}$	$\bar{6}'$. $(E'-\mathrm{i}D')(\bar{z}_1',z)_\mathrm{d}+D'(z,\bar{z}_1')_\mathrm{s}$
$7'$. $(A'+\mathrm{i})(z_1',z)_\mathrm{s}+(z,z_1')_\mathrm{d}$	$\bar{7}'$. $(E'-\mathrm{i}D')(\bar{z}_1',z)_\mathrm{s}+D'(z,\bar{z}_1')_\mathrm{d}$
$8'$. $B'(z,z_0)_\mathrm{s}$	$\bar{8}'$. $F'(z,\bar{z}_0)_\mathrm{s}$
$9'$. $B'(z,z_0)_\mathrm{s}$	$\bar{9}'$. $F'(z,\bar{z}_0)_\mathrm{s}$
$10'$. $B'(z,z_0)_\mathrm{d}$	$\overline{10}'$. $F'(z,\bar{z}_0)_\mathrm{d}$
11. $\mathrm{i}B'(z_0,z)_\mathrm{d}$	$\overline{11}$. $-\mathrm{i}F'(\bar{z}_0,z)_\mathrm{d}$
10. $\mathrm{i}B'(z_0,z)_\mathrm{d}+B'(z,z_0)_\mathrm{s}$	$\overline{10}$. $-\mathrm{i}F'(\bar{z}_0,z)_\mathrm{d}+F'(z,\bar{z}_0)_\mathrm{s}$
9. $\mathrm{i}B'(z_0,z)_\mathrm{s}+B'(z,z_0)_\mathrm{d}$	$\bar{9}$. $-\mathrm{i}F'(\bar{z}_0,z)_\mathrm{s}+F'(z,\bar{z}_0)_\mathrm{d}$
8. $B'(z,z_0)_\mathrm{d}$	$\bar{8}$. $F'(z,\bar{z}_0)_\mathrm{d}$
1. $C'(z,z_1)_\mathrm{s}$	$\bar{1}$. $G'(z,\bar{z}_1)_\mathrm{s}$
2. $C'(z,z_1)_\mathrm{d}$	$\bar{2}$. $G'(z,\bar{z}_1)_\mathrm{d}$
3. $C'(z,z_1)_\mathrm{d}+\mathrm{i}C'(z_1,z)_\mathrm{s}$	$\bar{3}$. $G'(z,\bar{z}_1)_\mathrm{d}-\mathrm{i}G'(\bar{z}_1,z)_\mathrm{s}$
4. $C'(z,z_1)_\mathrm{s}+\mathrm{i}C'(z_1,z)_\mathrm{d}$	$\bar{4}$. $G'(z,\bar{z}_1)_\mathrm{s}-\mathrm{i}G'(\bar{z}_1,z)_\mathrm{d}$
5. $-\mathrm{i}C'(z_1,z)_\mathrm{s}+C'(z,z_1)_\mathrm{d}$	$\bar{5}$. $\mathrm{i}G'(\bar{z}_1,z)_\mathrm{s}+G'(z,\bar{z}_1)_\mathrm{d}$
6. $C'(z,z_1)_\mathrm{d}$	$\bar{6}$. $G'(z,\bar{z}_1)_\mathrm{d}$
7. $C'(z,z_1)_\mathrm{s}$	$\bar{7}$. $G'(z,\bar{z}_1)_\mathrm{s}$

TABLE 3. APPROXIMATIONS NEAR z_0 AND THEIR REPRESENTATION NEAR OTHER TRANSITION POINTS

(Each entry is preceded by the label of the sector in which it is valid.)

$H = [z_1, z_0] = \delta^{-1}\exp\{\mathrm{i}q\}$	$I = -\mathrm{i}[z_0, z_1'] = -\mathrm{i}\delta^{-1}\exp\{-\mathrm{i}q\}$
$J = -\mathrm{i}[z_0, \bar{z}_0] = -\mathrm{i}\exp\{-2\mathrm{i}p-2\mathrm{i}q\}$	$K = [\bar{z}_0, z_0] = \exp\{2\mathrm{i}p+2\mathrm{i}q\}$
$L = H[\bar{z}_1, z_1] = \delta^{-1}\exp\{2\mathrm{i}p+\mathrm{i}q\}$	$M = I[z_1', \bar{z}_1'] = -\mathrm{i}\delta^{-1}\exp\{-2\mathrm{i}p-\mathrm{i}q\}$
10. $(z,z_0)_\mathrm{s}$	$\overline{10}$. $(J-\mathrm{i}K)(\bar{z}_0,z)_\mathrm{d}+K(z,\bar{z}_0)_\mathrm{s}$
11. $(z,z_0)_\mathrm{s}$	$\overline{11}$. $(J-\mathrm{i}K)(\bar{z}_0,z)_\mathrm{d}-\mathrm{i}J(z,\bar{z}_0)_\mathrm{s}$
$10'$. $-\mathrm{i}(z_0,z)_\mathrm{s}$	$\overline{10}'$. $(K+\mathrm{i}J)(z,\bar{z}_0)_\mathrm{d}+J(\bar{z}_0,z)_\mathrm{s}$
$9'$. $-\mathrm{i}(z_0,z)_\mathrm{d}$	$\bar{9}'$. $(K+\mathrm{i}J)(z,\bar{z}_0)_\mathrm{s}+J(\bar{z}_0,z)_\mathrm{d}$
$8'$. $-\mathrm{i}(z_0,z)_\mathrm{d}+(z,z_0)_\mathrm{s}$	$\bar{8}'$. $K(z,\bar{z}_0)_\mathrm{s}+J(\bar{z}_0,z)_\mathrm{d}$
8. $-\mathrm{i}(z_0,z)_\mathrm{s}+(z,z_0)_\mathrm{d}$	$\bar{8}$. $J(\bar{z}_0,z)_\mathrm{s}+K(z,\bar{z}_0)_\mathrm{d}$
9. $(z,z_0)_\mathrm{d}$	$\bar{9}$. $(J-\mathrm{i}K)(\bar{z}_0,z)_\mathrm{s}+K(z,\bar{z}_0)_\mathrm{d}$
1. $H(z,z_1)_\mathrm{s}$	$\bar{1}$. $L(z,\bar{z}_1)_\mathrm{s}$
2. $H(z,z_1)_\mathrm{d}$	$\bar{2}$. $L(z,\bar{z}_1)_\mathrm{d}$
3. $H(z,z_1)_\mathrm{d}+\mathrm{i}H(z_1,z)_\mathrm{s}$	$\bar{3}$. $L(z,\bar{z}_1)_\mathrm{d}-\mathrm{i}L(\bar{z}_1,z)_\mathrm{s}$
4. $H(z,z_1)_\mathrm{s}+\mathrm{i}H(z_1,z)_\mathrm{d}$	$\bar{4}$. $L(z,\bar{z}_1)_\mathrm{s}-\mathrm{i}L(\bar{z}_1,z)_\mathrm{d}$
5. $-\mathrm{i}H(z_1,z)_\mathrm{s}+H(z,z_1)_\mathrm{d}$	$\bar{5}$. $\mathrm{i}L(\bar{z}_1,z)_\mathrm{s}+L(z,\bar{z}_1)_\mathrm{d}$
6. $H(z,z_1)_\mathrm{d}$	$\bar{6}$. $L(z,\bar{z}_1)_\mathrm{d}$
7. $H(z,z_1)_\mathrm{s}$	$\bar{7}$. $L(z,\bar{z}_1)_\mathrm{s}$
$1'$. $I(z_1',z)_\mathrm{s}$	$\bar{1}'$. $M(\bar{z}_1',z)_\mathrm{s}$
$2'$. $I(z_1',z)_\mathrm{d}$	$\bar{2}'$. $M(\bar{z}_1',z)_\mathrm{d}$
$3'$. $I(z_1',z)_\mathrm{d}-\mathrm{i}I(z,z_1')_\mathrm{s}$	$\bar{3}'$. $M(\bar{z}_1',z)_\mathrm{d}+\mathrm{i}M(z,\bar{z}_1')_\mathrm{s}$
$4'$. $I(z_1',z)_\mathrm{s}-\mathrm{i}I(z,z_1')_\mathrm{d}$	$\bar{4}'$. $M(\bar{z}_1',z)_\mathrm{s}+\mathrm{i}M(z,\bar{z}_1')_\mathrm{d}$
$5'$. $\mathrm{i}I(z,z_1')_\mathrm{s}+I(z_1',z)_\mathrm{d}$	$\bar{5}'$. $-\mathrm{i}M(z,\bar{z}_1')_\mathrm{s}+M(\bar{z}_1',z)_\mathrm{d}$
$6'$. $I(z_1',z)_\mathrm{d}$	$\bar{6}'$. $M(\bar{z}_1',z)_\mathrm{d}$
$7'$. $I(z_1',z)_\mathrm{s}$	$\bar{7}'$. $M(\bar{z}_1',z)_\mathrm{s}$

of three unknowns, and this considerably simplifies their use in §5. To this end, we define δ, p and q by

$$\left.\begin{aligned}[0,\beta] &= \delta,\\ [0,z_0] &= \exp\{\mathrm{i}(p+q)\},\\ [\beta,z_1] &= \exp\{ip\},\end{aligned}\right\} \tag{3.11}$$

where β denotes the point where the anti-Stokes line joining z_1 and $\bar{z}_1$ cuts the real axis (see figure 1). Here δ, p and q are real because $[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}$ is real for z on the real axis between 0 and β, and 0 and z_0, and, β and z_1, are joined by anti-Stokes lines. Because $[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}$ is an even function of z

$$\left.\begin{aligned}[-\beta,0] &= [0,\beta],\\ [\bar{z}_0,0] &= [0,z_0],\\ \text{and}\qquad [\bar{z}_1',-\beta] &= [\beta,z_1].\end{aligned}\right\} \tag{3.12}$$

Also because the integral is pure imaginary

$$\overline{\int_\beta^{z_1}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,\mathrm{d}z} = -\int_\beta^{z_1}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,\mathrm{d}z,$$

i.e.

$$\int_\beta^{\bar{z}_1}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,\mathrm{d}z = -\int_\beta^{z_1}[(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\,\mathrm{d}z,$$

so that

$$\left.\begin{aligned}[\bar{z}_1,\beta] &= [\beta,z_1],\\ \text{and similarly}\qquad [\bar{z}_1',-\beta] &= [-\beta,z_1'].\end{aligned}\right\} \tag{3.13}$$

Use of Cauchy's theorem round a path $0\beta z_1 z_0 0$ and similar paths found by reflection in the real and imaginary axes, together with results (3.11)–(3.13), give

$$\left.\begin{aligned}[z_1,z_0] &= [\bar{z}_0,\bar{z}_1'] = \delta^{-1}\exp\{\mathrm{i}q\},\\ \text{and}\qquad [z_1',z_0] &= [\bar{z}_0,\bar{z}_1] = \delta\exp\{\mathrm{i}q\}.\end{aligned}\right\} \tag{3.14}$$

Results (3.11)–(3.14) allow the coefficients defined at the top of tables 1–3 to be written in terms of δ, p, and q, as shown.

The cuts shown in figure 1 may be inserted in any convenient way. Now that the W.K.B. approximations have been traced round each transition point, the cuts may be deformed onto adjacent anti-Stokes lines so that the sectors 4, $\bar{4}$, 4′, $\bar{4}'$, 10′ and $\overline{10}'$ disappear. This is done in §4 for convenience when presenting figure 2, and subsequent calculations from tables 1–3 are carried out with this adjustment.

4. The validity of the global phase-integral methods

The error analysis necessary to justify the procedure adopted in §3 is given by Olver (1974, chapter 6).

Let the domain Δ consist of the z-plane with the small circular regions $|z-(\pm z_k)| \leqslant \epsilon$, centred on the transition points, removed. Suppose further that Δ is rendered simply connected by cutting the z-plane from each transition point to infinity, as indicated in figure 2 by wavy lines. This ensures that

$$g(\alpha, z) = (z^2-\alpha^2)^3+1, \tag{4.1}$$

is regular and non-zero in Δ, and that the function

$$\xi(\mu,\alpha,z) = \mu\int g^{\frac{1}{2}}(\alpha,z)\,\mathrm{d}z, \tag{4.2}$$

is a regular function of z in Δ. It then follows from theorem (11.1) (Olver (1974), p. 222) that (2.5) has solutions $f_j(\mu,\alpha,z), j = 1,2$, regular in Δ, given by

$$f_j(\mu,\alpha,z) = g^{-\frac{1}{4}}(\alpha,z)\,\exp\{(-1)^{j-1}\xi(\mu,\alpha,z)\}\{1+\epsilon_j(\mu,\alpha,z)\}, \tag{4.3}$$

where
$$|\epsilon_j(\mu,\alpha,z)|,\left|\mu^{-1}g^{-\frac{1}{2}}\frac{\partial\epsilon_j(\mu,\alpha,z)}{\partial z}\right| \leqslant \exp\left\{\frac{1}{\mu}\int_{a_j}^{z}\left|\frac{1}{g^{\frac{1}{4}}}\frac{\partial^2}{\partial z^2}\left(\frac{1}{g^{\frac{1}{4}}}\right)\mathrm{d}z\right|\right\}-1, \tag{4.4}$$

provided that the reference point a_j may be joined to z by a ξ-progressive path in Δ, and that the integration in (4.4) is along this path.

The constant of integration in (4.2) may be arbitrarily assigned. Suppose that it is chosen to vanish at a transition point $z = \pm z_k$ so that, in the notation of §3,

$$g^{-\frac{1}{4}}(\alpha,z)\,\exp\{\xi(\mu,\alpha,z)\} = (\pm z_k, z). \tag{4.5}$$

If it can be shown that $|\epsilon_j(\mu,\alpha,z)| \ll 1$ in some subdomain of Δ, then the solutions $f_j(\mu,\alpha,z)$ are approximated by the W.K.B. approximations (2.10) in that subdomain.

Suppose, without loss of generality, that in (4.5) the transition point is chosen as z_1, and that the determination of $g^{\frac{1}{2}}(\alpha,z)$ is chosen that makes $\mathscr{R}\xi(\mu,\alpha,z) > 0$ in sectors 1 and 7 of figure 1. With the notation

$$\xi(\mu,\alpha,z) = u+\mathrm{i}v, \tag{4.6}$$

where u and v are real, the anti-Stokes lines through z_1, and hence those through $\bar{z}_1$, are the curves $u = 0$. Those through z_0 and $\bar{z}_0$, and, z_1' and $\bar{z}_1'$, are then from (3.14), $u = u_1 = -\ln\delta > 0$ and $u = 2u_1$ respectively. Because $\xi(\mu,\alpha,z)$ is a regular function of z in Δ, no two curves of the family $u =$ constant can meet in Δ. Furthermore, because u is a harmonic function, it has no extreme value at a point of Δ, and so no curve $u =$ constant is closed. The curve therefore terminates at infinity or on a boundary of Δ. Details of the curves $u =$ constant are given in figure 2, where it is now convenient to choose the cuts from the transition points along anti-Stokes lines as indicated by wavy lines.

For the solution $f_1(\mu,\alpha,z)$ of (4.3), suppose that the reference point a_1 in (4.4) is chosen at infinity on the positive real axis. It is seen immediately from figure 2, that a_1 may be joined to z by a ξ-progressive path in Δ, i.e. a path consisting of a finite chain of R_2 arcs for which u is non-decreasing, for any $z\in\Delta$ except in the sector between the anti-Stokes lines going to infinity from z_1, and its conjugate sector. There are also narrow shadow zones of width $O(\epsilon)$ caused by the small circles about the transition points, on either side of the cuts from z_0, $\bar{z}_0$, z_1' and $\bar{z}_1'$, and to the left of the anti-Stokes lines from z_1 and $\bar{z}_1$ going to infinity in the directions $\arg z = \frac{3}{8}\pi$ and $-\frac{3}{8}\pi$ respectively. Let Δ_1 denote the set of points z that may be reached from a_1 by a ξ-progressive path in Δ.

For points z on a ξ-progressive path sufficiently far away from the origin, $|z| \geqslant r_0$, say, the

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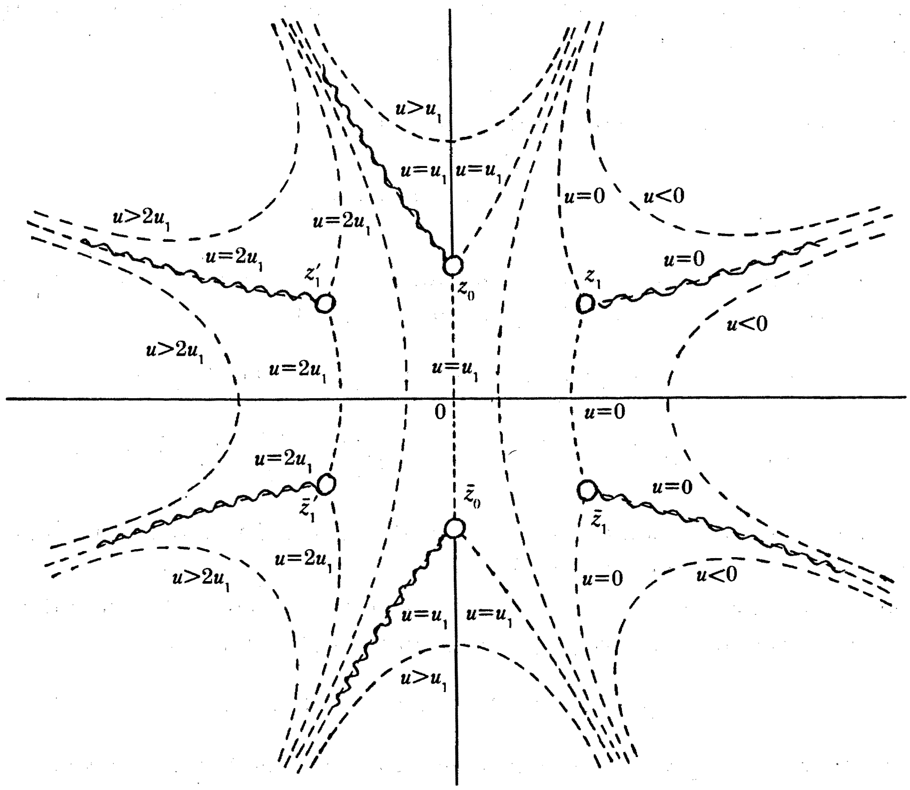

FIGURE 2. Lines: ----, curves u = constant; ∿∿∿, cuts.

paths may be chosen to make an angle no greater than an acute angle θ_0, say, with the radius vector from the origin. Hence, because from (4.1)

$$\frac{1}{g^{\frac{1}{4}}}\frac{\partial^2}{\partial z^2}\left(\frac{1}{g^{\frac{1}{4}}}\right)$$

is regular and bounded away from zero for $z \in \Delta$ such that $|z| \leqslant r_0$, and is $O(1/z^5)$ as $z \to \infty$,

$$\int_{a_1}^{z}\left|\frac{1}{g^{\frac{1}{4}}}\frac{\partial^2}{\partial z^2}\left(\frac{1}{g^{\frac{1}{4}}}\right)\mathrm{d}z\right| < -\int_{\infty}^{r_0} K_1 \frac{\mathrm{d}|z|\sec\theta_0}{|z|^5} + K_2 + \int_{r_0}^{\infty} K_3 \frac{\mathrm{d}|z|\sec\theta_0}{|z|^5},$$

$$< \frac{(K_1+K_3)\sec\theta_0}{4r_0^4} + K_2,$$

$$< K, \text{ say,}$$

where K, K_1, K_2, and K_3 are positive constants with $K_3 = 0$ if $|z| < r_0$ and $K_2 = K_3 = 0$ if z lies to the right of $\bar{z}_1 z_1$ with $|z| > r_0$. The right-hand side of (4.4) is thus less than $\exp(K/\mu) - 1$, giving $|\epsilon_1(\mu, \alpha, z)| < 0(1/\mu)$ as $\mu \to \infty$.

It follows that for sufficiently large μ, $f_1(\mu, \alpha, z)$ is approximated by that W.K.B. approximation associated with z_1, which was denoted by $(z_1, z)_\mathrm{d}$ in sectors 1 and 7 of figure 1, for $z \in \Delta_1$. This W.K.B. approximation is then a valid approximation to $f_1(\mu, \alpha, z)$ in sectors 8 and 9 associated with z_0, so that the attachment to z_0 used in building table 1, is justified. It is also noted that for the case $A = 0$ in table 1, the same W.K.B. approximation, which becomes subdominant in sectors 2 and 3 associated with z_1, is a valid approximation to $f_1(\mu, \alpha, z)$ as $z \to \infty$ in these sectors.

Because the choice of the constant of integration in (4.2) is arbitrary, z_0 can be used in place of z_1 in (4.5), resulting in all u-curves in figure 2 being relabelled by the addition of a constant $[z_0, z_1]$. The same argument then justifies the attachment to z_1' of the W.K.B. approximation associated with z_0. Similarly, the W.K.B. approximations to $f_1(\mu, \alpha, z)$ associated with $\bar{z}_1$ and $\bar{z}_0$ may be attached to $\bar{z}_0$ and $\bar{z}_1'$ respectively.

To deal with the other relevant choices of A in table 1, i.e. $A = \mathrm{i}$ and $A = -\mathrm{i}[z_1, \bar{z}_1]^2$ when the approximations listed are subdominant in sectors 4, 5, 6 and $\bar{4}$, $\bar{5}$, $\bar{6}$ respectively, the reference point a_1 may be chosen at $\infty \exp(\frac{1}{4}\mathrm{i}\pi)$ and its conjugate. Although Δ_1 is modified, it is unchanged to the left of $u = 0$ and the same validity results apply.

For the solution $f_2(\mu, \alpha, z)$ of (4.3), suppose that the reference point a_2 is chosen at infinity on the negative real axis, and let Δ_2 denote the set of points in Δ such that a_2 may be joined to z by a ξ-progressive path. On such a path u is now non-increasing, so that Δ_2 consists of Δ with the sectors between anti-Stokes lines going to infinity at z_1', $\bar{z}_1'$, z_0 and $\bar{z}_0$ removed, together with narrow shadow regions caused by the small circles about the transition points as before. In (4.4), $|\epsilon_2(\mu, \alpha, z)| < O(1/\mu)$ and $\mu \to \infty$, so that $f_2(\mu, \alpha, z)$ is approximated in Δ_2 by the W.K.B. approximation, which in sectors 1′ and 7′ of table 2 is denoted by $(z, z_1')_{\mathrm{d}}$. Because the approximation is valid in sectors 8′ and 9′ associated with z_0, the attachment to z_0 is justified. Similarly, the attachment to z_1, $\bar{z}_0$ and $\bar{z}_1$ of approximations associated with z_0, $\bar{z}_1'$ and $\bar{z}_0$, respectively, is valid. The case $A' = 0$ of table 2 is justified by this choice $a_2 = \infty \exp(\mathrm{i}\pi)$, and the other relevant cases $A' = -\mathrm{i}$ and $\mathrm{i}[\bar{z}_1', z_1']^2$, by the choices $a_2 = \infty \exp(\frac{3}{4}\pi\mathrm{i})$ and $\infty \exp(-\frac{3}{4}\mathrm{i}\pi)$ respectively.

Because the anti-Stokes lines joining the conjugate pairs $z_1, \bar{z}_1$; $z_0, \bar{z}_0$; and $z_1', \bar{z}_1'$; all lie in both Δ_1 and Δ_2, except for the segments lying within the small circles $|z - (\pm z_k)| \leqslant \epsilon$, the reattachment of W.K.B. approximations via these lines, carried out in the construction of tables 1 and 2, is justified.

All the global phase-integral methods used in the construction of tables 1 and 2 are thus justified. Furthermore, because the W.K.B. approximations used are valid throughout Δ_1 and Δ_2, they are valid in the neighbourhoods of all the transition points, and this provides a further check on the relevant entries in tables 1 and 2. It is also interesting to note for instance that if the cuts at z_1 and $\bar{z}_1$ were changed to run along the anti-Stokes lines going to infinity in the directions $\arg z = \pm\frac{3}{8}\pi$, the region of validity of the W.K.B. approximation to $f_1(\mu, \alpha, z)$, with a_1 chosen at infinity on the positive real axis, is all of Δ excluding only the narrow shadow regions either side of each cut.

To justify the results listed in table 3, a_1 may again be chosen at infinity on the positive real axis, but a_2 is chosen as $\infty \exp(\frac{1}{2}\mathrm{i}\pi)$. This restricts Δ_2 to the right half-plane together with the regions to the right of the cuts at z_0 and $\bar{z}_0$, but excluding the small circular regions about the transition points and the narrow shadow regions associated with them. In particular, there is a narrow shadow region to the right of the imaginary axis joining z_0 and $\bar{z}_0$. Both Δ_1 and Δ_2, however, include sectors 8 and $\bar{8}$ associated with z_0 and $\bar{z}_0$ respectively, and the attachment to $\bar{z}_0$ of the W.K.B. approximation associated with z_0 is justified. The reattachment from z_1 to $\bar{z}_1$ is valid as in previous cases. Finally the reattachment from z_0 to z_1' and from z_1' to $\bar{z}_1'$ involves the W.K.B. approximation associated with $f_1(\mu, \alpha, z)$ only, and is valid as before. This completes the validation of table 3, and the table obtained by conjugating table 3, is validated by a 'conjugate' argument.

5. Asymptotic estimates of the eigenvalues

Asymptotic estimates must first be found for the solutions (2.4) as $\zeta \to \pm\infty$. To obtain these from the information gained about $f(\mu, \alpha, z)$ in §3, the contour C must be chosen to satisfy (2.6). Because f and $\mathrm{d}f/\mathrm{d}z$ cannot have a common zero z (otherwise from (2.5) $f \equiv 0$), C must terminate at infinity. From tables 1–3, if f is subdominant in any one anti-Stokes sector at infinity, it is in general dominant in all other sectors at infinity. Hence although by selecting f appropriately, one end of C may be chosen arbitrarily at infinity, the other end must be chosen to terminate at infinity on an anti-Stokes line where f is oscillatory, and (2.6) satisfied by the factor $\exp(\mu z \zeta)$. Condition (2.6) can then be ensured for a restricted sector only of the ζ-plane, and the problem of relating solutions in different ζ-sectors arises. A similar difficulty and its resolution were encountered by Baldwin & Roberts (1972). The same method of resolution is used here.

Let R_N denote the ray

$$R_N = \{z \in \mathbf{C}: \arg z = \tfrac{1}{8}(2N-1)\,\pi \quad \text{for} \quad N \in \mathbf{Z}\}, \tag{5.1}$$

and D_N the anti-Stokes sector

$$D_N = \{z \in \mathbf{C}: \tfrac{1}{8}(2N-1)\,\pi < \arg z < \tfrac{1}{8}(2N+1)\,\pi \quad \text{for} \quad N \in \mathbf{Z}\}, \tag{5.2}$$

where $\mathbf{C}$ and $\mathbf{Z}$ denote the set of complex numbers and the set of integers respectively. The contour C may then approach infinity along R_N if

$$(2n - \tfrac{1}{4}N + \tfrac{5}{8})\,\pi < \arg \zeta < (2n - \tfrac{1}{4}N + \tfrac{13}{8})\,\pi \quad \text{for} \quad n \in \mathbf{Z}. \tag{5.3}$$

Let Γ_N denote a contour that starts at infinity on R_N and ends at infinity on R_{N+1}, and let $\gamma_{N,\,N+r}$ for $r = 1, 2, 3, 4, 5, 6$ denote a contour that starts at infinity on R_N and ends at infinity in the sector D_{N+r}. Then in the notation of (2.4), for $\arg \zeta$ satisfying (5.3), six linearly independent solutions of (1.1) with contours ending on R_N are given by $W(\mu\zeta, f_{N+r}; \gamma_{N,\,N+r})$, where f_{N+r} denotes that solution of (2.5) that is subdominant in D_{N+r}. Because $f_N(\mu, \alpha, z)$ is a regular function of z for all finite z, $f_N(\mu, \alpha, z\exp\{2\pi n\mathrm{i}\}) = f_N(\mu, \alpha, z)$ for $n \in \mathbf{Z}$, so that if N or r is changed by a multiple of 8, the solutions $W(\mu\zeta, f_{N+r}; \gamma_{N,\,N+r})$ are unchanged. Hence the subscripts for R_N, D_N, Γ_N, $\gamma_{N,\,N+r}$ and f_{N+r} are to be calculated modulo 8.

Define

$$\left.\begin{aligned} \mathbf{U}^{(N)}(\mu, \alpha, \zeta) &= \begin{bmatrix} u_1^{(N)}(\mu, \alpha, \zeta) \\ u_2^{(N)}(\mu, \alpha, \zeta) \\ u_3^{(N)}(\mu, \alpha, \zeta) \\ u_4^{(N)}(\mu, \alpha, \zeta) \\ u_5^{(N)}(\mu, \alpha, \zeta) \\ u_6^{(N)}(\mu, \alpha, \zeta) \end{bmatrix}, \\ \text{where} \qquad u_r^{(N)}(\mu, \alpha, \zeta) &= W(\mu\zeta, f_{N+r}; \gamma_{N,\,N+r}) \quad r = 1, 2, 3, \ldots, 6, \end{aligned}\right\} \tag{5.4}$$

for ζ satisfying (5.3).

If

$$(2n-\tfrac{1}{4}N+\tfrac{5}{8})\,\pi < \arg\zeta < (2n-\tfrac{1}{4}N+\tfrac{11}{8})\,\pi, \tag{5.5}$$

then the solution matrices $\mathbf{U}^{(N)}$ and $\mathbf{U}^{(N+1)}$ associated with the rays R_N and R_{N+1}, and defined according to (5.3), are both convergent, as is the solution $W(\mu\zeta, f; \Gamma_N)$, where f is any solution of (2.5). Now

$$W(\mu\zeta, f_N; \Gamma_N) \equiv 0,$$

because Γ_N may be deformed to infinity in D_N, and because (2.5) has only two independent solutions, all the solutions $W(\mu\zeta, f_j; \Gamma_N)$ for $j \neq N$ are proportional. This fact plays a fundamental role in relating the solutions $\mathbf{U}^{(N)}$ and $\mathbf{U}^{(N+1)}$ for ζ in (5.5) (cf. Baldwin & Roberts 1972).

From the definitions of Γ_N and $\gamma_{N,\,N+r}$,

$$\gamma_{N+1,\,N+1+r} = \gamma_{N,\,N+1+r} - \Gamma_N \quad \text{for} \quad r = 1, 2, 3, \ldots, 6,$$

so that

$$\begin{aligned} W(\mu\zeta, f_{N+r+1}; \gamma_{N+1,\,N+r+1}) &= W(\mu\zeta, f_{N+r+1}; \gamma_{N,\,N+r+1}) - W(\mu\zeta, f_{N+r+1}; \Gamma_N), \\ &= W(\mu\zeta, f_{N+r+1}; \gamma_{N,\,N+r+1}) - K_r(\mu, \alpha, N)\, W(\mu\zeta, f_{N+1}; \Gamma_N) \\ &\quad \text{for} \quad r = 1, 2, 3, \ldots, 6, \end{aligned} \tag{5.6}$$

where from above K_r is a constant of proportionality and therefore independent of ζ. Because f_{N+1} is subdominant in D_{N+1}, Γ_N may be chosen to terminate in D_{N+1} rather than on R_{N+1} for the solution $W(\mu\zeta, f_{N+1}; \Gamma_N)$, and then

$$W(\mu\zeta, f_{N+1}; \Gamma_N) = W(\mu\zeta, f_{N+1}; \gamma_{N,\,N+1}). \tag{5.7}$$

Also

$$W(\mu\zeta, f_{N+7}; \gamma_{N,\,N+7}) \equiv 0, \tag{5.8}$$

because $\gamma_{N,\,N+7}$ may be deformed to infinity in D_{N+7}. Hence the use of (5.6)–(5.8) with the definition (5.4) gives

$$\mathbf{U}^{(N+1)}(\mu, \alpha, \zeta) = T(\mu, \alpha, N)\, \mathbf{U}^{(N)}(\mu, \alpha, \zeta), \tag{5.9}$$

where

$$T(\mu, \alpha, N) = \begin{bmatrix} -K_1(\mu, \alpha, N) & 1 & 0 & 0 & 0 & 0 \\ -K_2(\mu, \alpha, N) & 0 & 1 & 0 & 0 & 0 \\ -K_3(\mu, \alpha, N) & 0 & 0 & 1 & 0 & 0 \\ -K_4(\mu, \alpha, N) & 0 & 0 & 0 & 1 & 0 \\ -K_5(\mu, \alpha, N) & 0 & 0 & 0 & 0 & 1 \\ -K_6(\mu, \alpha, N) & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{5.10}$$

and

$$K_r(\mu, \alpha, N) = W(\mu\zeta, f_{N+r+1}; \Gamma_N) / W(\mu\zeta, f_{N+1}; \Gamma_N). \tag{5.11}$$

From the results of §3 listed in tables 1–3, the vectors

$$\mathbf{K}(\mu,\alpha,N)=\begin{bmatrix}K_1(\mu,\alpha,N)\\K_2(\mu,\alpha,N)\\K_3(\mu,\alpha,N)\\K_4(\mu,\alpha,N)\\K_5(\mu,\alpha,N)\\K_6(\mu,\alpha,N)\end{bmatrix}, \tag{5.12}$$

may be estimated to leading order for large μ. Suppose that Γ_N in (5.11) is chosen to lie in D_N far from the origin, so that asymptotic estimates may be used for $f_{N+r+1}(\mu,\alpha,z)$ and $f_{N+1}(\mu,\alpha,z)$. Because, to leading order, the ratio $f_{N+r+1}(\mu,\alpha,z)/f_{N+1}(\mu,\alpha,z)$ is independent of z,

$$K_r(\mu,\alpha,N)\sim\frac{f_{N+r+1}(\mu,\alpha,z)}{f_{N+1}(\mu,\alpha,z)}\quad\text{as}\quad\mu\to\infty, \tag{5.13}$$

for sufficiently large z in D_N. To facilitate these calculations, the essential information contained in the tables 1–3 is shown in figures 3–10. Each figure shows the eight sectors D_N for $N=0,1,2,\ldots,7$ covering the z-plane. For large z, D_0 is the extension to infinity of the sectors $\overline{3}$, and $\overline{3}$, of figure 1. Similarly D_1 is associated with sectors 4, 5 and 6, D_2 with sectors 10, 11 and 10′, D_3 with sectors 4′, 5′ and 6′, D_4 with sectors 3′ and $\overline{3}'$, D_5 with sectors $\overline{4}'$, $\overline{5}'$ and $\overline{6}'$, D_6 with sectors $\overline{10}$, $\overline{11}$, and $\overline{10}'$ and D_7 with sectors $\overline{4}$, $\overline{5}$ and $\overline{6}$. In each sector is written a W.K.B. approximation. Figure $(j+3)$ shows the W.K.B. approximation to $f(\mu,\alpha,z)$ of (2.5) that is subdominant in D_j and its valid extension from the tables into all other sectors, for $j=0,1,2,\ldots,7$. For these results the cuts have been deformed onto anti-Stokes lines as described in the last paragraph of §3, and shown by wavy lines in figure 2. Because f_N is approximated by a W.K.B. approximation subdominant in D_N, (5.11)–(5.13) with figures 3–10 give

$$\mathbf{K}(\mu,\alpha,0)\sim\begin{bmatrix}-\mathrm{i}\delta^{-1}\exp\{\mathrm{i}q\}\\-\mathrm{i}\delta^{-2}\\-\mathrm{i}\delta^{-2}\\-\mathrm{i}\delta^{-2}\\-\mathrm{i}\delta^{-1}\exp\{-2\mathrm{i}p-\mathrm{i}q\}\\-\exp\{-4\mathrm{i}p\}\end{bmatrix},\quad \mathbf{K}(\mu,\alpha,1)\sim\begin{bmatrix}\delta^{-1}\exp\{-\mathrm{i}q\}\\\delta^{-1}\exp\{-\mathrm{i}q\}\\\delta^{-1}\exp\{-\mathrm{i}q\}\\\exp\{-2\mathrm{i}p-2\mathrm{i}q\}\\-\mathrm{i}\delta(\exp\{-4\mathrm{i}p\}+1)\exp\{-\mathrm{i}q\}\\-\mathrm{i}\delta\exp\{-\mathrm{i}q\}\end{bmatrix},$$

$$\mathbf{K}(\mu,\alpha,2)\sim\begin{bmatrix}1\\1\\2\delta\cos\{2p+2q\}\exp\{\mathrm{i}q\}\\-\mathrm{i}\exp\{2\mathrm{i}q\}\\-\mathrm{i}\exp\{2\mathrm{i}q\}\\-\mathrm{i}\exp\{2\mathrm{i}q\}\end{bmatrix},\quad \mathbf{K}(\mu,\alpha,3)\sim\begin{bmatrix}1+\exp\{4\mathrm{i}p\}\\\delta^{-1}\exp\{2\mathrm{i}p+\mathrm{i}q\}\\-\mathrm{i}\delta^{-2}\\-\mathrm{i}\delta^{-2}\\-\mathrm{i}\delta^{-2}\\-\delta^{-1}\exp\{-\mathrm{i}q\}\end{bmatrix},$$

$$\mathbf{K}(\mu,\alpha,4) \sim \begin{bmatrix} \delta^{-1}\exp\{-2\mathrm{i}p+\mathrm{i}q\} \\ -\mathrm{i}\delta^{-2}\exp\{-4\mathrm{i}p\} \\ -\mathrm{i}\delta^{-2}\exp\{-4\mathrm{i}p\} \\ -\mathrm{i}\delta^{-2}\exp\{-4\mathrm{i}p\} \\ -\delta^{-1}\exp\{-4\mathrm{i}p-\mathrm{i}q\} \\ -\exp\{-4\mathrm{i}p\} \end{bmatrix}, \quad \mathbf{K}(\mu,\alpha,5) \sim \begin{bmatrix} -\mathrm{i}\delta^{-1}\exp\{-2\mathrm{i}p-\mathrm{i}q\} \\ -\mathrm{i}\delta^{-1}\exp\{-2\mathrm{i}p-\mathrm{i}q\} \\ -\mathrm{i}\delta^{-1}\exp\{-2\mathrm{i}p-\mathrm{i}q\} \\ -\exp\{-2\mathrm{i}p-2\mathrm{i}q\} \\ -2\delta\cos 2p\exp\{-\mathrm{i}q\} \\ -\delta\exp\{2\mathrm{i}p-\mathrm{i}q\} \end{bmatrix},$$

$$\mathbf{K}(\mu,\alpha,6) \sim \begin{bmatrix} 1 \\ 1 \\ -2\mathrm{i}\delta\cos\{2p+2q\}\exp\{2\mathrm{i}p+\mathrm{i}q\} \\ -\mathrm{i}\exp\{4\mathrm{i}p+2\mathrm{i}q\} \\ -\mathrm{i}\exp\{4\mathrm{i}p+2\mathrm{i}q\} \\ -\mathrm{i}\exp\{4\mathrm{i}p+2\mathrm{i}q\} \end{bmatrix}, \quad \mathbf{K}(\mu,\alpha,7) \sim \begin{bmatrix} 1+\exp\{4\mathrm{i}p\} \\ -\mathrm{i}\delta^{-1}\exp\{4\mathrm{i}p+\mathrm{i}q\} \\ -\mathrm{i}\delta^{-2}\exp\{4\mathrm{i}p\} \\ -\mathrm{i}\delta^{-2}\exp\{4\mathrm{i}p\} \\ -\mathrm{i}\delta^{-2}\exp\{4\mathrm{i}p\} \\ -\mathrm{i}\delta^{-1}\exp\{2\mathrm{i}p-\mathrm{i}q\} \end{bmatrix}. \tag{5.14}$$

The solutions of (1.1) are regular functions of ζ for all finite ζ, hence the solutions $\mathbf{U}^{(N)}(\mu,\alpha,z)$, represented by Laplace integrals as shown by (2.4) and (5.4) in the range (5.3), may be defined throughout the z-plane by analytic continuation. Hence from (5.4), $\mathbf{U}^{(N+8)}(\mu,\alpha,z) = \mathbf{U}^{(N)}(\mu,\alpha,z)$, so that from (5.9)

$$\prod_{j=0}^{7} T(\mu,\alpha,N+7-j) = I, \tag{5.15}$$

for any integer N where I denotes the sixth-order unit matrix. An attempt to use (5.15) to check the calculation of $T(\mu,\alpha,N)$ when the leading approximations (5.14) were used in (5.10), failed. This is because δ, defined in (3.11), is exponentially small for large μ, so that many of the elements in (5.14) are exponentially large. Although the leading-order terms of the matrix product in (5.15) cancel, the remaining terms are not sufficiently small to approximate the unit matrix. The calculation of higher-order terms in (5.14) appears to be difficult. The inclusion of further terms with the W.K.B. approximations used to construct tables 1–3 gives no immediate improvement, and more accurate estimates from (5.11) are required. This loss of accuracy suggests that repeated use of (5.9) using leading approximations to $T(\mu,\alpha,N)$ may not be adequate to relate solutions as $\zeta\to\pm\infty$. It will be seen below, however, that one application only of (5.9) is required and that the leading order approximations are adequate.

Consideration of (5.3) with the choice $n=0$ shows that $\arg\zeta = 0$ lies within the range for $N = 3$, 4, 5 and 6, and that $\arg\zeta = \pi$ lies within the range for $N = -1$, 0, 1 and 2. A single application only of (5.9) is thus required for $N = 2$ if $\mathbf{U}^{(3)}$ is used to describe the solutions for $\arg\zeta = 0$ and $\mathbf{U}^{(2)}$ for $\arg\zeta=\pi$. Alternatively, the choice $N=6$ may be used if $\mathbf{U}^{(6)}$ and $\mathbf{U}^{(7)}$ are used to describe the solutions when $\arg\zeta = 0$ and $\arg\zeta=\pi$ respectively.

It remains to estimate the solutions $\mathbf{U}^{(2)}$ and $\mathbf{U}^{(3)}$, or $\mathbf{U}^{(6)}$ and $\mathbf{U}^{(7)}$, for $\arg\zeta = 0$ and

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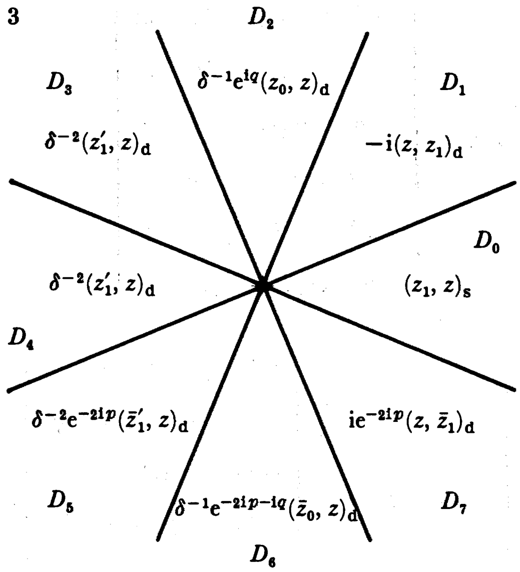

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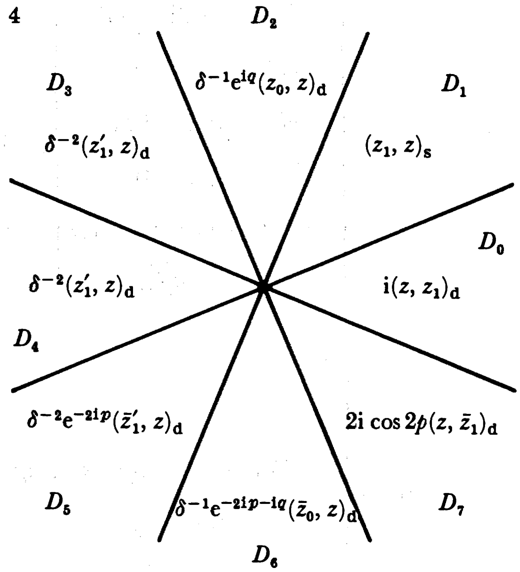

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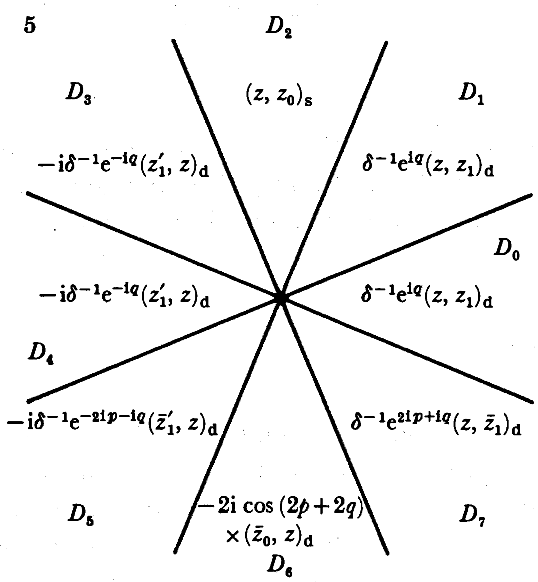

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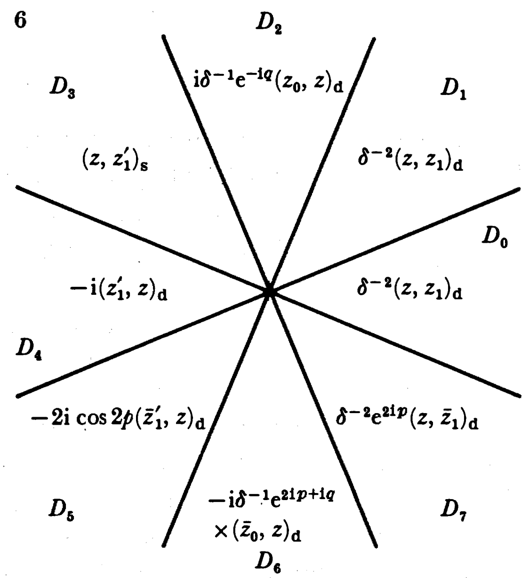

FIGURE 3. Subdominancy in D_0.
FIGURE 4. Subdominancy in D_1.
FIGURE 5. Subdominancy in D_2.
FIGURE 6. Subdominancy in D_3.

$\arg\zeta = \pi$, respectively, as $|\zeta| \to \infty$. The elements of these vectors are, from (2.4) and (5.4), approximated in the various sectors D_N by integrals proportional to

$$\int_C \frac{1}{[(z^2-\alpha^2)^3+1]^{\frac{1}{4}}} \exp\left\{\mu\left(z\zeta + \int_{\pm z_k}^{z} [(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\, dz\right)\right\}. \tag{5.16}$$

A steepest-descent analysis for large μ is appropriate. There are saddle points at

$$z = [\alpha^2 + (\zeta^2-1)^{\frac{1}{3}}]^{\frac{1}{2}}.$$

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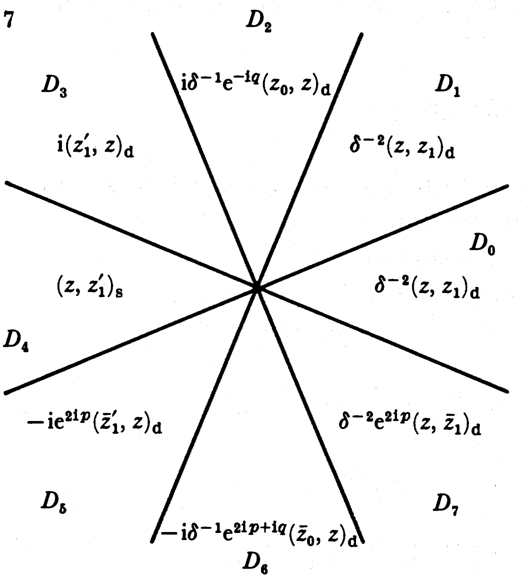

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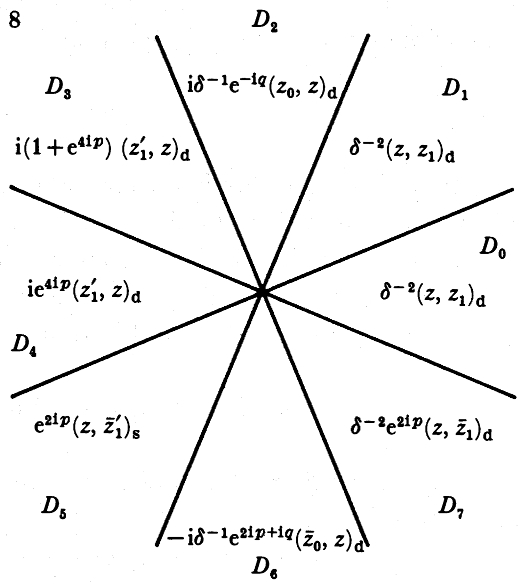

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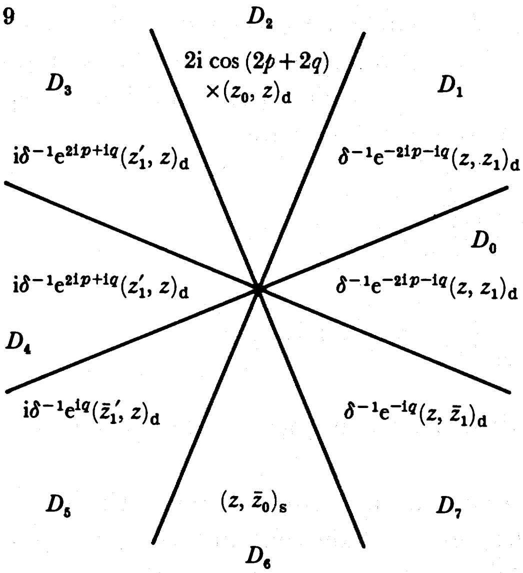

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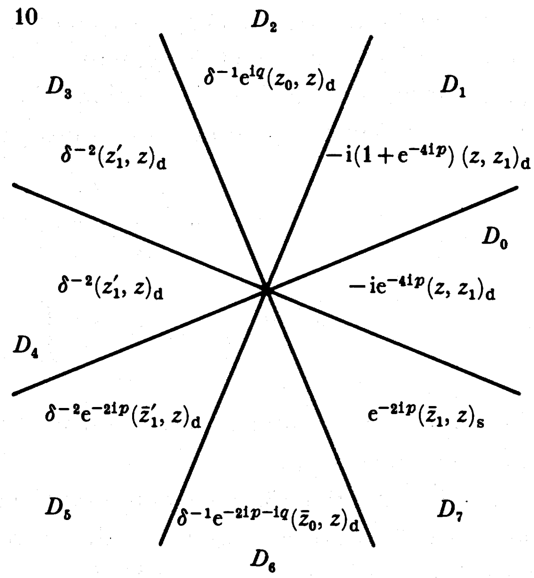

FIGURE 7. Subdominancy in D_4.
FIGURE 8. Subdominancy in D_5.
FIGURE 9. Subdominancy in D_6.
FIGURE 10. Subdominancy in D_7.

For large ζ, these are located asymptotically at

$$z = s_k = \zeta^{\frac{1}{3}} \exp\left(\tfrac{1}{3}k\pi\mathrm{i}\right) \quad \text{for} \quad k = 0, 1, 2, \ldots, 5, \tag{5.17}$$

and the curves of steepest descent near s_k are given by

$$\arg(z - s_k) = -\tfrac{1}{3}\arg\zeta + \tfrac{1}{6}k\pi + n\pi \quad \text{for integral } n. \tag{5.18}$$

If the contour C of (5.16) is deformed to pass over s_k along a steepest path given in (5.18), the contribution to (5.16) is proportional to

$$w_k = \zeta^{-\frac{1}{6}} \exp\{\tfrac{3}{4}\mu\zeta^{\frac{4}{3}}\, e^{\frac{1}{3}k\pi i}\}. \tag{5.19}$$

When $\arg\zeta = 0$, w_k is dominant for $k = 0, 1, 5$, and subdominant for $k = 2, 3, 4$, and when $\arg\zeta = \pi$, w_k is dominant for $k = 1, 2, 3$, and subdominant for $k = 0, 4, 5$.

For those sectors in which the solution under consideration has $[(z^2-\alpha^2)^3+1]^{\frac{1}{2}} \sim -z^3$ as $z \to \infty$, the steepest curves are those associated with $\exp(z\zeta - \frac{1}{4}z^4)$, i.e. $\mathscr{I}\{z\zeta - \frac{1}{4}z^4\} = \mathscr{I}\{\frac{3}{4}\zeta^{\frac{4}{3}} \exp(\frac{1}{3}k\pi i)\}$ for $k = 0, 2$, and 4 from (5.17). If $z = r\exp(i\theta)$ they satisfy

$$r|\zeta| \sin(\theta + \arg\zeta) - \tfrac{1}{4}r^4 \sin 4\theta = \tfrac{3}{4}|\zeta|^{\frac{4}{3}} \sin(\tfrac{4}{3}\arg\zeta + \tfrac{1}{3}k\pi), \tag{5.20}$$

for $k = 0, 2, 4$. Similarly for those sectors in which $[(z^2-\alpha^2)^3+1]^{\frac{1}{2}} \sim z^3$ as $z \to \infty$, the steepest curves satisfy

$$r|\zeta| \sin(\theta + \arg\zeta) + \tfrac{1}{4}r^4 \sin 4\theta = \tfrac{3}{4}|\zeta|^{\frac{4}{3}} \sin(\tfrac{4}{3}\arg\zeta + \tfrac{1}{3}k\pi), \tag{5.21}$$

for $k = 1, 3, 5$. These curves are shown for $\arg\zeta = 0$ and $\arg\zeta = \pi$ in figures 11 and 12. In each case $\theta = 0, \pi$ is a steepest curve, and there is symmetry about the real axis. Local information for the saddle points is given by (5.18). Arrows in both figures point down a steepest gradient. It should be noted that equations (5.20)–(5.21) have been deduced on the assumption that z is large but this is not true along the whole real axis, which it is claimed is a steepest curve. The relevant equations with approximation for large ζ only are

$$\mathscr{I}\left\{z\zeta + \int_{\pm z_k}^{z} [(z^2+\zeta^2)^3+1]^{\frac{1}{2}}\, dz\right\}_{z=\zeta^{\frac{1}{3}}\exp(\frac{1}{3}k\pi i)}^{z} = 0,$$

i.e.
$$\mathscr{I}\left\{z\zeta - \zeta^{\frac{4}{3}}\exp(\tfrac{1}{3}k\pi i) + \int_{\zeta^{\frac{1}{3}}\exp(\frac{1}{3}k\pi i)}^{z} [(z^2-\alpha^2)^3+1]^{\frac{1}{2}}\, dz\right\} = 0,$$

for $k = 0, 1, 2, \ldots, 5$, and these are satisfied for all real z when $\arg\zeta = 0$ with $k = 0, 3$, and $\arg\zeta = \pi$ with $k = 2, 5$.

From the definition (5.4) the solution $u_r^{(N)}$ has its integrand in (5.16) subdominant in D_{N+r} and from figures 3–10 is dominant in all other sectors through which the contour $C \equiv \gamma_{N,N+r}$ passes. Hence from (5.2) if $N+r$ is even, $[(z^2-\alpha^2)^3+1]^{\frac{1}{2}} \sim -z^3$ in $D_{N+r}, D_{N+r\pm1}, D_{N+r\pm3}, \ldots$ and $[(z^2-\alpha^2)^3+1]^{\frac{1}{2}} \sim z^3$ in $D_{N+r\pm2}, D_{N+r\pm4}, \ldots$, the signs being reversed if $N+r$ is odd.

Consider the solution matrix $\mathbf{U}^{(3)}$ for $\arg\zeta = 0$. For $u_1^{(3)}$ the contour is deformed to pass down R_3 until it meets the curve of steepest descent over s_2. It then passes over s_2 and goes to infinity in D_4 via the steepest curve (see figure 11^-). For $u_2^{(3)}$ the contour is deformed down R_3, over s_2 in D_3 (figure 11^-) and then to infinity in D_5 over s_3 (figure 11^+) via curves of steepest descent. Similar paths can be found for the remaining components of $\mathbf{U}^{(3)}$ from figure 11, and similar paths for the components of $\mathbf{U}^{(2)}$ for $\arg\zeta = \pi$ from figure 12. The important information required is the list of saddle points traversed by the contour for each solution, and this is shown in table 4. The leading-order contributions to the solutions from the saddle points is given by (5.19), and from the statements following (5.19) the only solutions from $\mathbf{U}^{(3)}$ when $\arg\zeta = 0$

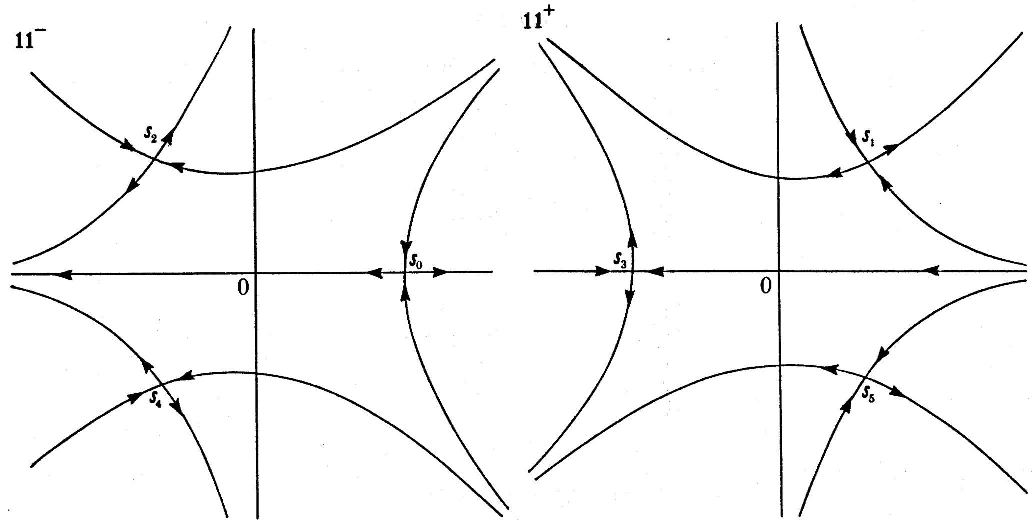

FIGURE 11⁻. Steepest curves associated with $\exp(z\zeta - \frac{1}{4}z^4)$; $(\arg \zeta = 0)$.
FIGURE 11⁺. Steepest curves associated with $\exp(z\zeta + \frac{1}{4}z^4)$; $(\arg \zeta = 0)$.

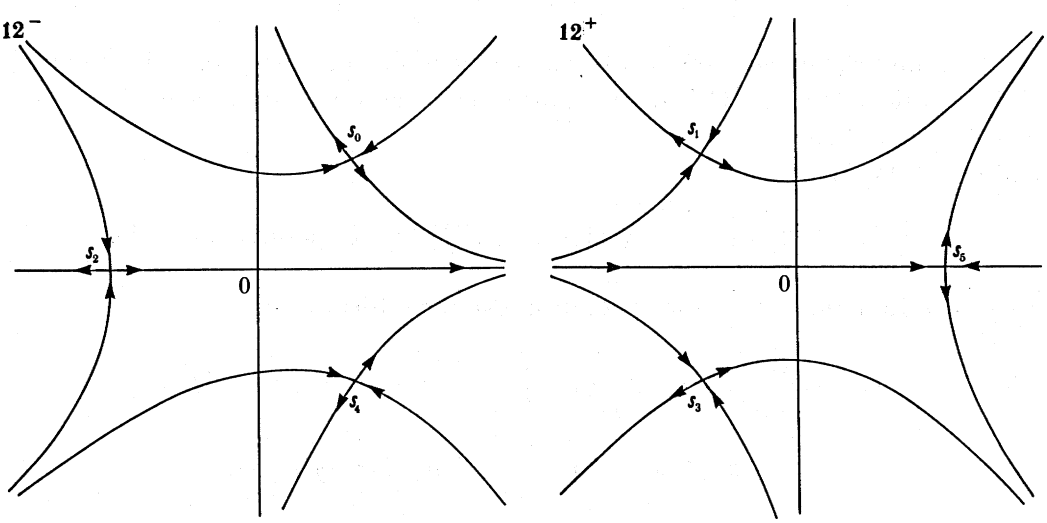

FIGURE 12⁻. Steepest curves associated with $\exp(z\zeta - \frac{1}{4}z^4)$; $(\arg \zeta = \pi)$.
FIGURE 12⁺. Steepest curves associated with $\exp(z\zeta + \frac{1}{4}z^4)$; $(\arg \zeta = \pi)$.

and $\mathbf{U}^{(2)}$ when $\arg \zeta = \pi$ that will satisfy (1.2) are $u_r^{(3)}$ for $r = 1$, 2, and 3, and $u_r^{(2)}$ for $r = 4$, 5, and 6. Now from (5.9)–(5.10)

$$u_1^{(3)} = -K_1(\mu, \alpha, 2)\, u_1^{(2)} + u_2^{(2)},$$

$$u_2^{(3)} = -K_2(\mu, \alpha, 2)\, u_1^{(2)} + u_3^{(2)},$$

$$u_3^{(3)} = -K_3(\mu, \alpha, 2)\, u_1^{(2)} + u_4^{(2)},$$

and hence the only solution to the boundary-value problem (1.1) with (1.2) for large μ, is provided by $u_3^{(3)}$ or $u_4^{(2)}$ if the parameters α and μ are chosen to satisfy

$$K_3(\mu, \alpha, 2) = 0. \tag{5.22}$$

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TABLE 4

solution	saddle points traversed
$u_1^{(3)}$	s_2
$u_2^{(3)}$	s_2, s_3
$u_3^{(3)}$	s_2, s_3, s_4
$u_4^{(3)}$	s_2, s_3, s_4, s_5
$u_5^{(3)}$	s_0, s_2, s_3
$u_6^{(3)}$	s_1
$u_1^{(2)}$	s_1
$u_2^{(2)}$	s_0, s_2, s_5
$u_3^{(2)}$	s_0, s_3, s_4, s_5
$u_4^{(2)}$	s_0, s_4, s_5
$u_5^{(2)}$	s_0, s_5
$u_6^{(2)}$	s_0

From (5.14) this gives

$$\cos(2p+2q) = 0, \tag{5.23}$$

because $\delta \neq 0$, and (3.2) and (3.11) then give

$$\cos\left\{2\mu\int_0^{\sqrt{(1-\alpha^2)}} \sqrt{[1-(\alpha^2+y^2)^3]}\,\mathrm{d}y\right\} = 0, \tag{5.24}$$

because the contour of integration may be chosen along the imaginary axis with $z = \mathrm{i}y$. If the alternative solutions $\mathbf{U}^{(6)}$ and $\mathbf{U}^{(7)}$ are used, the eigenvalue condition becomes $K_3(\mu, \alpha, 6) = 0$, which again gives (5.23).

Condition (5.24) gives

$$\mu = (2n-1)\,\pi\Big/4\int_0^{\sqrt{(1-\alpha^2)}} \sqrt{[1-(\alpha^2+y^2)^3]}\,\mathrm{d}y \quad \text{for positive integral } n, \tag{5.25}$$

because $\mu > 0$. Equations (2.1) and (2.2) then give the critical values

$$R_c = \min_{0<\alpha<1}\left\{(2n-1)\,\pi\Big/4\sqrt{\alpha}\int_0^{\sqrt{(1-\alpha^2)}} \sqrt{[1-(\alpha^2+y^2)^3]}\,\mathrm{d}y\right\}^4, \tag{5.26}$$

$$a_c = (2n-1)\,\pi\alpha_c\Big/4\int_0^{\sqrt{(1-\alpha_c^2)}} \sqrt{[1-(\alpha_c^2+y^2)^3]}\,\mathrm{d}y, \tag{5.27}$$

where α_c is the value of α that minimizes (5.26).

Numerical integration with Simpson's rule yields

$$\max_{0<\alpha<1} \sqrt{\alpha}\int_0^{\sqrt{(1-\alpha^2)}} \sqrt{[1-(\alpha^2+y^2)^3]}\,\mathrm{d}y = 0.5343,$$

when

$$\alpha = \alpha_c = 0.514,$$

and (5.26)–(5.27) then give the following estimates for R_c and a_c.

n	R_c	a_c
1	4.67	0.54
2	3.78×10^2	1.62
3	2.92×10^3	2.71
4	1.12×10^4	3.79
5	3.06×10^4	4.87
6	6.83×10^4	5.96

Also as $n \to \infty$, $R_c \sim 74.7n^4$ and $a_c \sim 1.1n$. Clearly the large-parameter assumptions made to obtain these results are not valid for the lower modes. It will be seen in §6, however, that apart from the lowest they are surprisingly good, and even the lowest-mode estimates are of the right order, and provide a good starting point for a numerical search. Attempts to improve the estimates (5.26) and (5.27) require an improved estimate (5.23) from (5.22). This leads back to the difficulties associated with (5.15) described earlier.

6. Numerical results

Because (1.1) involves even-ordered derivatives, and ζ occurs in the coefficients as ζ^2 only, the eigensolutions of (1.1) with (1.2) are either even or odd functions of ζ. The problems of obtaining numerical results direct from (1.1) are thus considerably eased by working on the interval $(0, \infty)$ with the initial conditions

$$DW = D^3W = D^5W = 0 \quad \text{when } \zeta = 0 \text{ for even modes,} \tag{6.1}$$

and

$$W = D^2W = D^4W = 0 \quad \text{when } \zeta = 0 \text{ for odd modes.} \tag{6.2}$$

The problem is then of similar form to that solved by Baldwin (1987). The compound matrix method (see, for example, Drazin & Reid (1981), pp. 311–317) may again be used with one change i.e. $\zeta \to \zeta^2$ in the coefficient matrix, and similar minor adjustments in the boundary conditions. It is again convenient to integrate in the reverse direction from $\zeta \approx 10$ to $\zeta = 0$.

Starting from the asymptotic estimates given at the end of §5, the following results, believed to be correct to the number of figures shown, were obtained.

n	R_c	a_c
1	9.781 365 67	0.726 05
2	411.720 155	1.679 1
3	3006.709 534	2.737 9
4	11 382.695 328	3.813 0
5	30 916.253 4	4.891 6
6	68 778.117	5.971

The odd-numbered modes were found to be even in ζ and the even-numbered modes odd.

The lowest mode's estimates were inaccurate as expected, but were sufficiently accurate to give rapid convergence to the results obtained. The second mode estimates for R_c and a_c have errors of about 40% and 5% respectively, and thereafter steadily improve.

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Appendix

A description is given here of a method by which the Stokes and anti-Stokes lines defined in §2 may be determined in the first quadrant of the plane.

It is first noted that the transition points (2.7) all lie on the curve $\mathscr{C}$ defined by

$$z^2 = \alpha^2 - \exp(-\mathrm{i}t) \quad \text{for real } t, \tag{A 1a}$$

or if $z = r\exp(\mathrm{i}\theta)$,

$$r^4 - 2\alpha^2 r^2 \cos 2\theta + \alpha^4 - 1 = 0. \tag{A 1b}$$

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This curve has symmetry about both the real and imaginary axes, and for α in the range (2.3) consists of a single loop enclosing the origin. At z_1, the angle between the radius vector from the origin and the positive direction of the tangent to $\mathscr{C}$ is $\pi-\theta_0$, where

$$\theta_0 = \arctan\left[(\alpha^2+2)/\alpha^2\surd 3\right].$$

Hence the angles between the negative tangent to $\mathscr{C}$ at z_1, and the Stokes lines given by (2.12) are

$$\theta_0-\tfrac{4}{3}\theta_1+\frac{4\pi}{9},\quad \theta_0-\tfrac{4}{3}\theta_1+\frac{10\pi}{9}\quad\text{and}\quad \theta_0-\tfrac{4}{3}\theta_1+\frac{16\pi}{9},$$

where
$$\theta_1 = \tfrac{1}{2}\arctan\left[\surd 3/(2\alpha^2+1)\right].$$

It may be shown that $\theta_0-\frac{4}{3}\theta_1$ is a monotonic increasing function of α in (2.3) so that

$$\frac{2\pi}{9} < \theta_0-\tfrac{4}{3}\theta_1 < \frac{5\pi}{18}.$$

Thus only one of the Stokes lines leaves z_1 inside $\mathscr{C}$, and hence two of the anti-Stokes lines leave z_1 inside $\mathscr{C}$. Whether any of these lines subsequently cross $\mathscr{C}$ may be deduced by considering the integral

$$I = \int_{z_1}^{z} \left[(z^2-\alpha^2)^3+1\right]^{\frac{1}{2}}\,\mathrm{d}z, \tag{A 2}$$

along $\mathscr{C}$, and by using the definitions (2.11) and (2.13) together with Cauchy's theorem.

To make the integrand of (A 2) precise, a cut must be placed at each transition point. Suppose that these are chosen as indicated in figure 1 by wavy lines, which all lie outside $\mathscr{C}$. Without loss of generality, that determination that is positive when $z = 0$ is chosen for the following argument.

Making the substitution (A 1*a*) in (A 2), followed by $t = \pi+u$, gives

$$I = \frac{\mathrm{i}}{\surd 2}\int_{-\frac{1}{3}\pi}^{u} \frac{\surd[\cos(\frac{3}{2}u)]}{(\alpha^4+2\alpha^2\cos u+1)^{\frac{1}{4}}}\exp(-\mathrm{i}\phi)\,\mathrm{d}u,$$

for $|u| < \arccos(\alpha^2)$, where positive roots are extracted, and

$$\phi = \tfrac{7}{4}u-\tfrac{1}{2}\arctan\left[\sin u/(\alpha^2+\cos u)\right].$$

Hence for α in (2.3) and $|u| \leqslant \frac{1}{3}\pi$, $|\phi| \leqslant \frac{1}{2}\pi$ so that $\mathscr{R}I \geqslant 0$ and $\mathscr{I}I > 0$ with equality if and only if $u = \frac{1}{3}\pi$. Because $u = \frac{1}{3}\pi$ corresponds to $\bar{z}_1$ on $\mathscr{C}$, no Stokes or anti-Stokes line from z_1 can cross $\mathscr{C}$ between z_1 and $\bar{z}_1$, otherwise Cauchy's theorem applied to the integrand of (A 2) round a closed contour starting from z_1, running along $\mathscr{C}$ and returning via a Stokes or anti-Stokes line, is violated. There remains the possibility, however, that an anti-Stokes line joins z_1 and $\bar{z}_1$.

For t of (A 1*a*) in the range $0 \leqslant t \leqslant \frac{1}{3}\pi$, (A 2) may be written

$$I = \frac{1}{\surd 2}\int_{\frac{2}{3}\pi}^{t} \frac{\surd[\sin(\frac{3}{2}t)]}{(\alpha^4-2\alpha^2\cos t+1)^{\frac{1}{4}}}\exp(-\mathrm{i}\psi)\,\mathrm{d}t,$$

where
$$\psi = \tfrac{7}{4}t-\tfrac{3}{4}\pi+\tfrac{1}{2}\arctan\frac{\sin t}{\alpha^2-\cos t}+\begin{cases}0\\ \frac{1}{2}\pi,\end{cases}$$

according as $\cos t \lessgtr \alpha^2$. Here ψ is a monotonic increasing function of α in (2.3) and hence

$$-\tfrac{1}{2}\pi < \psi < \tfrac{7}{12}\pi,$$

with $\frac{1}{2}\pi < \psi < \frac{7}{12}\pi$ when $t = \frac{2}{3}\pi$. It follows that as t decreases from $\frac{2}{3}\pi$ to 0, $\mathscr{R}I$ increases from a negative to a positive value, so that one anti-Stokes line from z_1 crosses $\mathscr{C}$ between z_1 and z_0. By changing the lower limit of the integral from $\frac{2}{3}\pi$ to 0, it also follows that no anti-Stokes line from z_0 crosses $\mathscr{C}$ between z_0 and z_1. Because two anti-Stokes lines from z_1 lie inside $\mathscr{C}$, one remains inside $\mathscr{C}$, and because it cannot cross the imaginary axis that is an anti-Stokes line inside $\mathscr{C}$, it must cross the real axis and by symmetry terminate at $\bar{z}_1$. The remaining two anti-Stokes line from z_1, and the anti-Stokes line from z_0 in the first quadrant, must therefore go to infinity in the first quadrant. Because they cannot cross, the anti-Stokes line from z_0, and that from z_1 crossing $\mathscr{C}$ must go to infinity in the direction $\arg z = \frac{3}{8}\pi$, the remaining line from z_1 going to infinity in the direction $\arg z = \frac{1}{8}\pi$, as the two lines from z_1 going to infinity are separated by a Stokes line that must go to infinity in the direction $\arg z = \frac{1}{4}\pi$. The structure of the anti-Stokes lines is thus fixed for α in (2.3), and by use of symmetry are as shown in figure 1.

As no Stokes line leaving z_1 crosses $\mathscr{C}$ between z_1 and $\bar{z}_1$ or terminates at $\bar{z}_1$, the second Stokes line leaving z_1 outside $\mathscr{C}$ must go to infinity along and above the positive real axis.

It remains to determine the path of the Stokes line leaving z_1 inside $\mathscr{C}$, and the Stokes line leaving z_0 in the first quadrant, which also starts inside $\mathscr{C}$. Because these paths are not relevant to the analysis required for this paper, all further details are omitted, and figure 1 completed for sufficiently small values of α in (2.3).

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